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Estimating the Spin–Independent WIMP–Nucleon Coupling from Direct Dark Matter Detection Data

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Abstract

Weakly Interacting Massive Particles (WIMPs) are one of the leading candidates for Dark Matter. For understanding the nature of WIMPs and identifying them among new particles produced at colliders (hopefully in the near future), determinations of their mass and couplings on nucleons from direct Dark Matter detection experiments are essential. Based on our model-independent method for determining the WIMP mass from experimental data, I present a way to also estimate the spin-independent (SI) WIMP–nucleon coupling by using measured recoil energies directly. This method is independent of the velocity distribution of halo WIMPs as well as (practically) of the as yet unknown WIMP mass. In a background-free environment, for a WIMP mass of ~ 100 GeV the SI WIMP–nucleon coupling could in principle be estimated with an uncertainty of $\sim 15\%$ by using 2 (or 3) $\times 50$ events from experiments.

1 Introduction

Astronomical observations and measurements indicate that more than 80% of all matter in the Universe is dark (i.e., interacts at most very weakly with electromagnetic radiation and ordinary matter). The dominant component of this cosmological Dark Matter must be due to some yet to be discovered, non-baryonic particles. Weakly Interacting Massive Particles (WIMPs) χ arising in several extensions of the Standard Model of electroweak interactions are one of the leading candidates for Dark Matter. WIMPs are stable particles with masses roughly between 10 GeV and a few TeV and interact with ordinary matter only weakly (for reviews, see Refs. [1, 2]).

Currently, the most promising method to detect different WIMP candidates is the direct detection of the recoil energy deposited in a low-background underground detector by elastic scattering of ambient WIMPs off target nuclei [3, 4]. The recoil energy spectrum can be calculated from an integral over the one-dimensional velocity distribution function of halo WIMPs, $f_1(v)$, where v is the absolute value of the WIMP velocity in the laboratory frame. In our earlier work [5], we presented a way to reconstruct this one-dimensional velocity distribution function and to estimate its moments from the recoil spectrum as well as from measured recoil energies *directly* in direct Dark Matter detection experiments. *Neither* the WIMP-nucleus scattering cross section *nor* the local WIMP density is required in this analysis.

However, the mass of halo WIMPs is needed for the reconstruction of the (moments of the) WIMP velocity distribution. Therefore, as the next step we developed a model-independent method based on the reconstruction of the moments of $f_1(v)$ for determining the WIMP mass m_χ by combining two sets of (future) experimental data with different target nuclei directly [6, 7]. To do so, one simply requires that the values of a given moment of $f_1(v)$ estimated by both experiments agree. This leads to a simple expression for determining m_χ , which can be solved analytically and each moment can be used. Moreover, by assuming that the ratio of the spin-independent (SI) scattering cross sections on protons and on neutrons is known, an additional expression for determining m_χ has been derived. By combining the estimators for different moments with each other and with the estimator derived by making the assumption about the ratio of the SI cross sections, one can yield the best estimate of the WIMP mass [7]. Here we found again that neither a prior knowledge about the WIMP-nucleus cross section nor that about the local WIMP density is required.

Meanwhile, in the second method for the determination of the WIMP mass, the product of the local WIMP density times the SI WIMP-proton cross section, $\rho_0 \sigma_{\chi p}^{\text{SI}}$, appearing in the expression for the scattering spectrum cancels out when we use the identity of this product for two different targets. Hence, as will be shown in the paper, once the WIMP mass can be determined one could then use this information to estimate $\sigma_{\chi p}^{\text{SI}}$ conversely. Remind that, in order to identify new particles produced at e.g., the Large Hadron Collider (LHC) to be indeed WIMPs detected by direct detection [8], estimates of or constraints on their mass and couplings on nucleons from direct detection experiments are essential. However, due to the degeneracy between ρ_0 and $\sigma_{\chi p}^{\text{SI}}$, for estimating the SI WIMP cross section by this method one has to make an assumption for the local WIMP density, which can so far be estimated with an uncertainty of a factor of ~ 2 [1, 2]. Nevertheless, our simulations show that, in spite of the large statistical uncertainty due to very few events, for a WIMP mass of ~ 100 GeV, $\sigma_{\chi p}^{\text{SI}}$ could be estimated with an uncertainty of 30% by using 2 (or 3) $\times 50$ events from experiments. This result is (much) better than our estimate of the local Dark Matter density.

The remainder of this article is organized as follows. In Sec. 2 I discuss the possibility of constraining the WIMP mass and its coupling on nucleons from a single experiment. In Sec. 3 I present the method for estimating the spin-independent WIMP-nucleon coupling by combining

two (or more) experiments. Some numerical results based on Monte Carlo simulations of future experiments will also be presented. In Sec. 4 the analysis will be extended to the case of spin-dependent (SD) WIMP–nucleon couplings. I conclude in Sec. 5. Some technical details for our analysis will be given in an appendix.

2 Constraining the SI WIMP–nucleon coupling

The basic expression for the differential event rate for elastic WIMP–nucleus scattering is given by [1]:

$$\frac{dR}{dQ} = \mathcal{A} F^2(Q) \int_{v_{\min}}^{v_{\max}} \left[\frac{f_1(v)}{v} \right] dv. \quad (1)$$

Here R is the direct detection event rate, i.e., the number of events per unit time and unit mass of detector material, Q is the energy deposited in the detector, $F(Q)$ is the elastic nuclear form factor, $f_1(v)$ is the one-dimensional velocity distribution function of the WIMPs impinging on the detector, v is the absolute value of the WIMP velocity in the laboratory frame. The constant coefficient \mathcal{A} is defined as

$$\mathcal{A} \equiv \frac{\rho_0 \sigma_0}{2m_\chi m_{\text{r},\text{N}}^2}, \quad (2)$$

where ρ_0 is the WIMP density near the Earth and σ_0 is the total cross section ignoring the form factor suppression. The reduced mass $m_{\text{r},\text{N}}$ is defined by

$$m_{\text{r},\text{N}} \equiv \frac{m_\chi m_{\text{N}}}{m_\chi + m_{\text{N}}}, \quad (3)$$

where m_χ is the WIMP mass and m_{N} that of the target nucleus. Finally, v_{\min} is the minimal incoming velocity of incident WIMPs that can deposit the energy Q in the detector:

$$v_{\min} = \alpha \sqrt{Q}, \quad (4)$$

with the transformation constant

$$\alpha \equiv \sqrt{\frac{m_{\text{N}}}{2m_{\text{r},\text{N}}^2}}, \quad (5)$$

and v_{\max} is the maximal WIMP velocity in the Earth’s reference frame, which is related to the escape velocity from our Galaxy at the position of the Solar system, $v_{\text{esc}} \gtrsim 600$ km/s.

The local WIMP density at the position of the Solar system, ρ_0 , appearing in the expression (1) for the scattering event rate has conventionally been determined by means of the measurement of the rotation curve of our Galaxy. The currently most commonly used value for ρ_0 is [1, 2]

$$\rho_0 \approx 0.3 \text{ GeV/cm}^3. \quad (6)$$

However, as mentioned in the introduction, due to our location inside the Milky Way, it is more difficult to measure the accurate rotation curve of our own Galaxy than those of other galaxies; an uncertainty of a factor of ~ 2 has thus usually been adopted [1, 2]¹:

$$\rho_0 = 0.2 - 0.8 \text{ GeV/cm}^3. \quad (10)$$

¹Recently, some new techniques have been developed for determining ρ_0 with a higher precision [9, 10, 11, 12, 13]. These estimates give rather *larger* values for ρ_0 ; e.g., Catena and Ullio gave [9]

$$\rho_0 = 0.39 \pm 0.03 \text{ GeV/cm}^3, \quad (7)$$

On the other hand, in most theoretical models, the spin-independent WIMP interaction on nucleus with an atomic mass number $A \gtrsim 30$ dominates over the spin-dependent (SD) interaction [1, 2]. Additionally, for the lightest supersymmetric neutralino, which is perhaps the best motivated WIMP candidate [1, 2, 15], and for all WIMPs which interact primarily through Higgs exchange, the SI scalar coupling is approximately the same on both protons p and neutrons n [16]. The “pointlike” cross section σ_0 in Eq. (2) can thus be written as

$$\begin{aligned}\sigma_0^{\text{SI}} &= \left(\frac{4}{\pi}\right) m_{\text{r},\text{N}}^2 [Z f_p + (A - Z) f_n]^2 \\ &\simeq \left(\frac{4}{\pi}\right) m_{\text{r},\text{N}}^2 A^2 |f_p|^2 \\ &= A^2 \left(\frac{m_{\text{r},\text{N}}}{m_{\text{r},\text{p}}}\right)^2 \sigma_{\text{xp}}^{\text{SI}},\end{aligned}\tag{11}$$

and the SI WIMP cross section on protons (nucleons) can be given as

$$\sigma_{\text{xp}}^{\text{SI}} = \left(\frac{4}{\pi}\right) m_{\text{r},\text{p}}^2 |f_p|^2,\tag{12}$$

where $f_{\text{p}(\text{n})}$ are the effective $\chi\chi\text{pp}(\text{nn})$ four-point couplings, A is the atomic mass number of the target nucleus, and $m_{\text{r},\text{p}}$ is the reduced mass of the WIMP mass m_χ and the proton mass m_p . Here the tiny mass difference between a proton and a neutron has been neglected.

As mentioned in the introduction, in our earlier work it has been found that one could in principle determine m_χ from direct detection experiments with neither a prior knowledge of σ_0 nor that of ρ_0 [6, 7]. Conversely, I will show in this article that one could also estimate or at least constrain the WIMP–nucleon cross section from experimental data directly *without* knowing m_χ , but for this estimation an assumption about ρ_0 is needed.

2.1 Expression for estimating the SI WIMP–nucleon coupling

Our analysis starts from the expression (1) for the event rate for the elastic WIMP–nucleus scattering directly. By using a time-averaged recoil spectrum, and assuming that no directional information exists, the normalized one-dimensional velocity distribution function of halo WIMPs, $f_1(v)$, has been solved from Eq. (1) analytically [5] and, consequently, its generalized moments can be estimated by [5, 7]²

$$\langle v^n \rangle(v(Q_{\text{min}}), v(Q_{\text{max}})) = \int_{v(Q_{\text{min}})}^{v(Q_{\text{max}})} v^n f_1(v) dv$$

and Salucci *et al.* even gave [11]

$$\rho_0 = 0.43 \pm 0.11 \pm 0.10 \text{ GeV/cm}^3.\tag{8}$$

Moreover, instead of a spherical symmetric density profile assumed in Refs. [9, 11], in Refs. [10, 12, 13] the authors considered an axisymmetric density profile for a flattened Galactic Dark Matter halo [14] caused by the disk structure of the luminous baryonic component. It was found that the local density of such a non-spherical Dark Matter halo could be enhanced by $\sim 20\%$ or larger [10, 12] and Pato *et al.* gave therefore [12]

$$\rho_0 = 0.466 \pm 0.033(\text{stat}) \pm 0.077(\text{syst}) \text{ GeV/cm}^3.\tag{9}$$

²Here we have implicitly assumed that Q_{max} is so large that terms involving $-2Q_{\text{max}}^{(n+1)/2} r(Q_{\text{max}})/F^2(Q_{\text{max}})$ are negligible. Due to sizable contributions from large recoil energies [5], this is not necessarily true, especially for some not-very-high Q_{max} in the experimental reality, and/or heavy detector targets, and/or heavy WIMPs. Nevertheless, considering the large statistical uncertainties due to (very) few events in the highest energy ranges, this should practically be a good approximation.

$$= \alpha^n \left[\frac{2Q_{\min}^{(n+1)/2} r(Q_{\min})/F^2(Q_{\min}) + (n+1)I_n(Q_{\min}, Q_{\max})}{2Q_{\min}^{1/2} r(Q_{\min})/F^2(Q_{\min}) + I_0(Q_{\min}, Q_{\max})} \right]. \quad (13)$$

Here $v(Q) = \alpha\sqrt{Q}$, $Q_{(\min, \max)}$ are the experimental minimal and maximal cut-off energies of the data set, respectively,

$$r(Q_{\min}) \equiv \left(\frac{dR}{dQ} \right)_{\text{expt}, Q=Q_{\min}} \quad (14)$$

is an estimated value of the *measured* recoil spectrum $(dR/dQ)_{\text{expt}}$ (*before* normalized by an experimental exposure, \mathcal{E}) at $Q = Q_{\min}$, and $I_n(Q_{\min}, Q_{\max})$ can be estimated through the sum:

$$I_n(Q_{\min}, Q_{\max}) = \sum_{a=1}^{N_{\text{tot}}} \frac{Q_a^{(n-1)/2}}{F^2(Q_a)}, \quad (15)$$

where the sum runs over all events in the data set that satisfy $Q_a \in [Q_{\min}, Q_{\max}]$ and N_{tot} is the number of such events. Note that, firstly, by using the second line of Eq. (13) $\langle v^n \rangle(v(Q_{\min}), v(Q_{\max}))$ can be determined independently of the local WIMP density ρ_0 , of the velocity distribution function of incident WIMPs, $f_1(v)$, as well as of the WIMP–nucleus cross section σ_0 . Secondly, $r(Q_{\min})$ and $I_n(Q_{\min}, Q_{\max})$ are two key quantities for our analysis, which can be estimated either from a functional form of the recoil spectrum or from experimental data (i.e., the measured recoil energies) directly³.

By substituting the second expression in Eq. (11) into Eq. (1), and using the fact that the integral over the one-dimensional WIMP velocity distribution on the right-hand side of Eq. (1) is the minus-first moment of this distribution, which can be estimated by Eq. (13) with $n = -1$, we have

$$\begin{aligned} \left(\frac{dR}{dQ} \right)_{\text{expt}, Q=Q_{\min}} &= \mathcal{E} \mathcal{A} F^2(Q_{\min}) \int_{v(Q_{\min})}^{v(Q_{\max})} \left[\frac{f_1(v)}{v} \right] dv \\ &= \mathcal{E} \left(\frac{2\rho_0 A^2 |f_p|^2}{\pi m_\chi} \right) F^2(Q_{\min}) \cdot \frac{1}{\alpha} \left[\frac{2r(Q_{\min})/F^2(Q_{\min})}{2Q_{\min}^{1/2} r(Q_{\min})/F^2(Q_{\min}) + I_0} \right]. \end{aligned} \quad (16)$$

Using the definition (5) of α , the *squared* SI WIMP coupling on protons (nucleons) can be expressed as [17]

$$|f_p|^2 = \frac{1}{\rho_0} \left[\frac{\pi}{4\sqrt{2}} \left(\frac{1}{\mathcal{E} A^2 \sqrt{m_N}} \right) \right] \left[\frac{2Q_{\min}^{1/2} r(Q_{\min})}{F^2(Q_{\min})} + I_0 \right] (m_\chi + m_N). \quad (17)$$

Note that, firstly, the experimental exposure \mathcal{E} appearing in the denominator relates the *actual* counting rate $(dR/dQ)_{\text{expt}}$ to the normalized rate in Eq. (1). Secondly, due to the neglect of the terms $-2Q_{\max}^{1/2} r(Q_{\max})/F^2(Q_{\max})$ and $-2r(Q_{\max})/F^2(Q_{\max})$ in the denominator and numerator of the expression (13) for $\langle v^n \rangle$, respectively, $|f_p|^2$ determined by Eq. (17) would be *overestimated*, since the contributions from the two neglected terms are *negative* and the former is much larger than the later. However, because $|f_p|^2$ estimated by Eq. (17) is inversely proportional to the local WIMP density, whose commonly used value would possibly be *underestimated* (see Eqs. (6) to (10)), one should therefore at least be able to give an *upper* bound on $|f_p|^2$. Then, by using the

³All formulae needed for estimating $r(Q_{\min})$, $I_n(Q_{\min}, Q_{\max})$, and their statistical errors are given in the appendix.

standard Gaussian error propagation, the statistical uncertainty on $|f_p|^2$ estimated by Eq. (17) can be given as

$$\sigma(|f_p|^2) = |f_p|^2 \left[\frac{\sigma^2(m_\chi)}{(m_\chi + m_N)^2} + \mathcal{N}_m^2 \sigma^2(1/\mathcal{N}_m) + \frac{2\mathcal{N}_m \text{cov}(m_\chi, 1/\mathcal{N}_m)}{(m_\chi + m_N)} \right]^{1/2}, \quad (18)$$

where I have used [5]

$$\mathcal{N}_m^{-1} = \frac{2Q_{\min}^{1/2} r(Q_{\min})}{F^2(Q_{\min})} + I_0. \quad (19)$$

2.2 From a single experiment

The expression (17) for estimating the (squared) SI WIMP–proton coupling depends on three quantities: $r(Q_{\min})$, I_0 , and the WIMP mass m_χ . As argued in Ref. [7], from a *single* recoil spectrum one *cannot* estimate m_χ *without* making some assumptions about the velocity distribution $f_1(v)$. Hence, as a model-independent analysis, one could only express/constrain $|f_p|^2$ as a (linear) function/interval of the WIMP mass on the coupling–mass plane by using Eq. (17) with a *single* experiment. Meanwhile, from Eqs. (1) and (2), it can be found that, due to the degeneracy between the local WIMP density ρ_0 and the WIMP–nucleus cross section σ_0 , one *cannot* estimate both of them independently⁴. Thus, for using Eq. (17), the simplest way is making an assumption for the local WIMP density ρ_0 .

In Fig. 1 I show the simulated results for a ⁷⁶Ge target with 5,000 experiments based on the Monte Carlo method⁵. The theoretical predicted recoil spectrum for the shifted Maxwellian velocity distribution [1, 2, 5] with a Sun’s orbital velocity in the Galactic frame $v_0 = 220$ km/s, an Earth’s velocity in the Galactic frame $v_e = 1.05 v_0$,⁶ and a maximal cut-off velocity of the velocity distribution function $v_{\max} = 700$ km/s, as well as the commonly used elastic nuclear form factor for the SI cross section [20, 1, 2]:

$$F_{\text{SI}}^2(Q) = \left[\frac{3j_1(qR_1)}{qR_1} \right]^2 e^{-(qs)^2} \quad (20)$$

have been used. The SI WIMP–proton cross section has been set as 10^{-8} pb. The commonly used value of $\rho_0 = 0.3$ GeV/cm³ has been used for both predicting the recoil spectrum and analyzing generated events. The experimental maximal cut-off energy Q_{\max} has been set as 50 keV and the threshold energy has been assumed to be negligible. Each experiment contains an expected number of 50 total events; the actual event number is Poisson-distributed around this expectation value. The mass of incident WIMPs has been chosen as 25 (dotted magenta), 100 (dashed blue), and 300 (double-dashed black) GeV, respectively.

As we can see here, the prefactor, i.e., the slope of the linear function $|f_p|^2(m_\chi)$, in Eq. (17) is obviously *underestimated*. For the case of an input WIMP mass of 300 GeV, the theoretical value of $|f_p|^2$ (the filled green square) is even outside the 1σ statistical uncertainty interval. This is because that the experimental maximal cut-off energy has been set as only 50 keV here. Remind that it is usually assumed that the WIMP flux on the Earth is negligible at velocities

⁴In contrast, as I will show in Sec. 4, the ratios between different WIMP–nucleon couplings/cross sections can be determined *without* knowing the mass and the local WIMP density [18, 17, 19].

⁵Note that, rather than the mean values, in this article we give always the median values of the reconstructed results from the simulated experiments.

⁶The time dependence of the Earth’s velocity in the Galactic frame [1, 2] has been ignored.

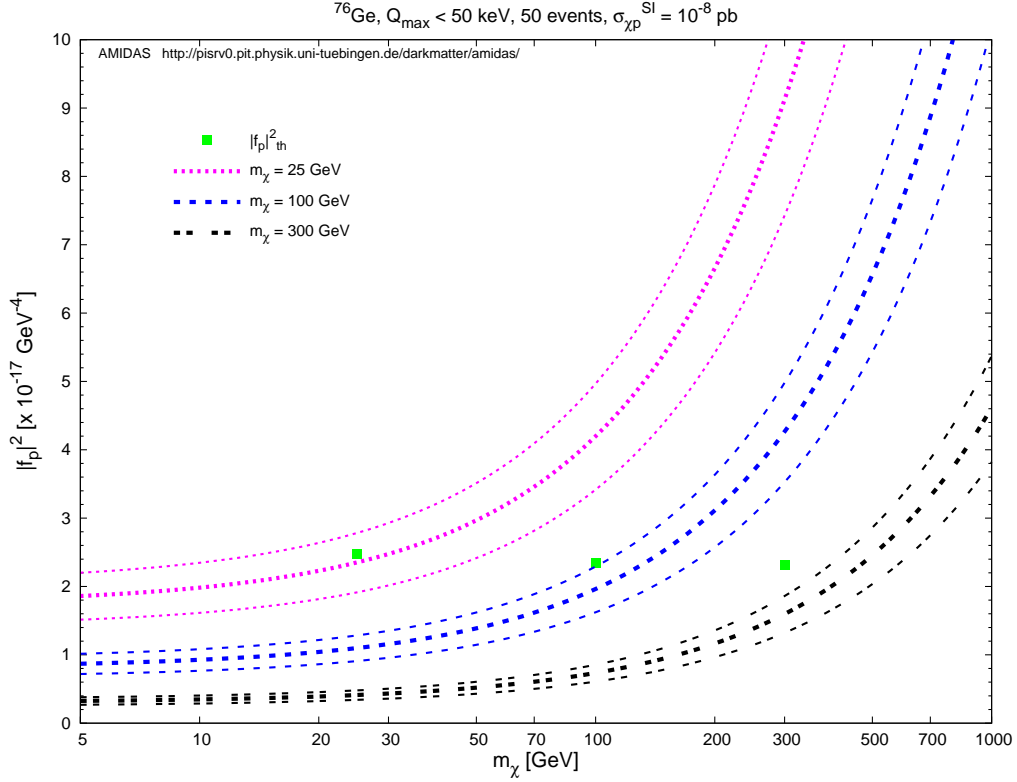


Figure 1: The *squared* SI WIMP–proton couplings $|f_p|^2$ estimated by Eq. (17) and the lower and upper bounds of their 1σ statistical uncertainties as functions of the WIMP mass for a ^{76}Ge target. The theoretical predicted recoil spectrum for the shifted Maxwellian velocity distribution with $v_0 = 220$ km/s, $v_e = 1.05 v_0$, and $v_{\text{max}} = 700$ km/s as well as the commonly used elastic nuclear form factor for the SI cross section given in Eq. (20) have been used. The SI WIMP–proton cross section has been set as 10^{-8} pb. The experimental maximal cut–off energy Q_{max} has been set as 50 keV and the threshold energy has been assumed to be negligible. Each experiment contains 50 total events on average. The mass of incident WIMPs has been chosen as 25 (dotted magenta), 100 (dashed blue), and 300 (double–dashed black) GeV, respectively. The filled green squares indicate the input WIMP masses and the theoretical values of $|f_p|^2$. See the text for further details.

exceeding the maximal velocity v_{max} . This leads thus to a kinematic maximum of the recoil energy

$$Q_{\text{max,kin}} = \frac{v_{\text{max}}^2}{\alpha^2}. \quad (21)$$

For a WIMP mass of 100 (300) GeV, this kinematic maximum for a Ge target is 264 (504) keV. Hence, I_0 in the prefactor of the linear function $|f_p|^2(m_\chi)$ given in Eq. (17) has been (strongly) underestimated. In Fig. 2 we increase therefore the maximal cut–off energy Q_{max} to 100 keV. It can be seen clearly that, by extending the detector sensitivity to higher energy ranges, the underestimated I_0 and thereby the prefactor of the linear function $|f_p|^2(m_\chi)$ can be corrected significantly⁷.

⁷Remind that, since we neglected the term $-2Q_{\text{max}}^{1/2}r(Q_{\text{max}})/F^2(Q_{\text{max}})$ in the second bracket in Eq. (17), which contributes negatively, all results shown in this paper are somehow *overestimated*.

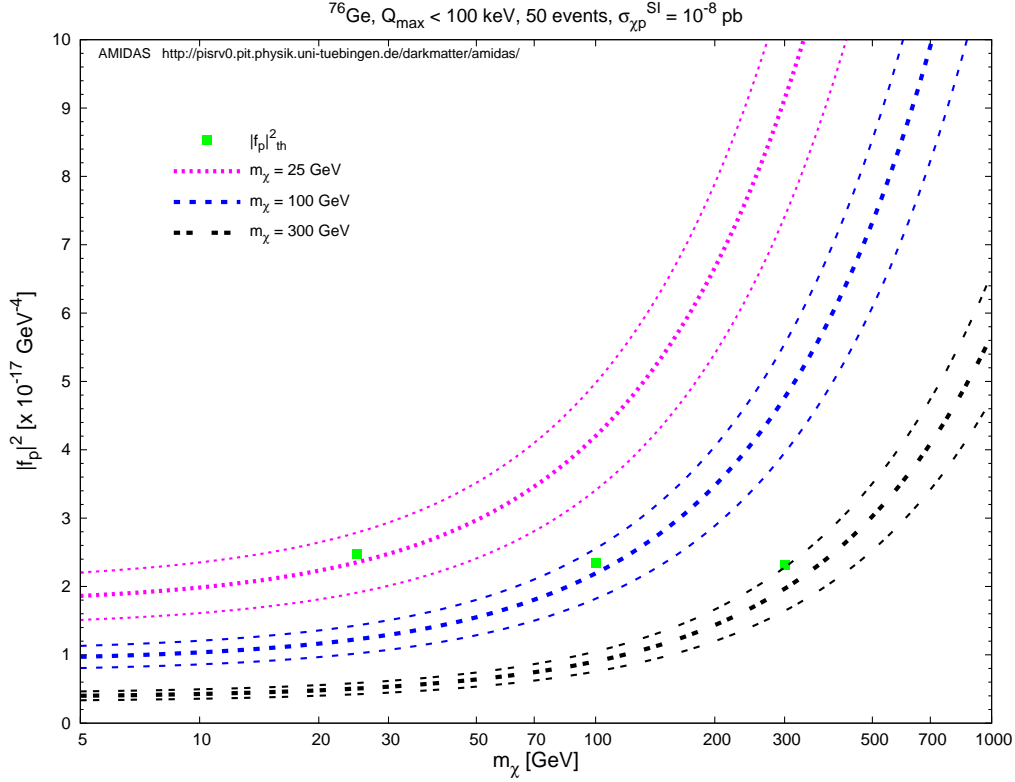


Figure 2: As in Fig. 1, except that the maximal cut-off energy Q_{\max} has been increased to 100 keV.

On the other hand, by substituting Eq. (17) into Eq. (12), one can express the SI WIMP–proton (nucleon) cross section as a function of the WIMP mass, $\sigma_{\chi p}^{\text{SI}}(m_\chi)$, on the cross section versus WIMP mass plane. In Figs. 3 I show the simulated results for a Ge target with an input WIMP mass of 100 GeV. The experimental minimal cut-off energy Q_{\min} has been set as 0 (upper) and 5 (lower) keV. As a comparison I show also four extra curves drawn conventionally by using the shifted Maxwellian velocity distribution with four Sun’s orbital velocities: $v_0 = 180$ km/s (dash-double-dotted orange), $v_0 = 200$ km/s (dash-dotted cyan), $v_0 = 220$ km/s (dotted magenta), $v_0 = 240$ km/s (double-dotted black), and the form factor given in Eq. (20).

As shown here, two results analyzed by Eq. (17) and by the conventional method with an assumed halo model are compatible with each other in the mass and cross section ranges around and higher than the input values, whereas in the low WIMP mass range, these two curves show a significant incompatibility. Hence, by comparing results from these two analyses, one could in principle – for the first step with only one experiment observing positive signals – give the *lower* bounds of the WIMP mass and its cross section on protons (nucleons) ($m_\chi \gtrsim 40$ GeV and $\sigma_{\chi p}^{\text{SI}} \gtrsim 7 \times 10^{-9}$ pb from the upper frame of Figs. 3 in our simulation) from a *single* experiment. Moreover, the lower frame of Figs. 3 shows that, due to the *non-negligible* threshold energy the conventional method is (much) more unsensitive for lighter WIMPs (in contrast, the uncertainty interval given by Eqs. (17) and (18) becomes only a bit wider) and the curves thus go sharply upwards as the WIMP mass decreases. The incompatibility between two analyses becomes larger and one could therefore even give more strict constraints on the WIMP mass and the SI cross section ($m_\chi \gtrsim 45$ GeV and $\sigma_{\chi p}^{\text{SI}} \gtrsim 7.5 \times 10^{-9}$ pb in our simulation).

In Figs. 4 we examine the same comparison of two analyses for a rather light input WIMP mass of 25 GeV. The lower frame shows that, with the non-negligible threshold energy one could even give the *upper* bounds of the WIMP mass and its cross section on protons (nucleons)

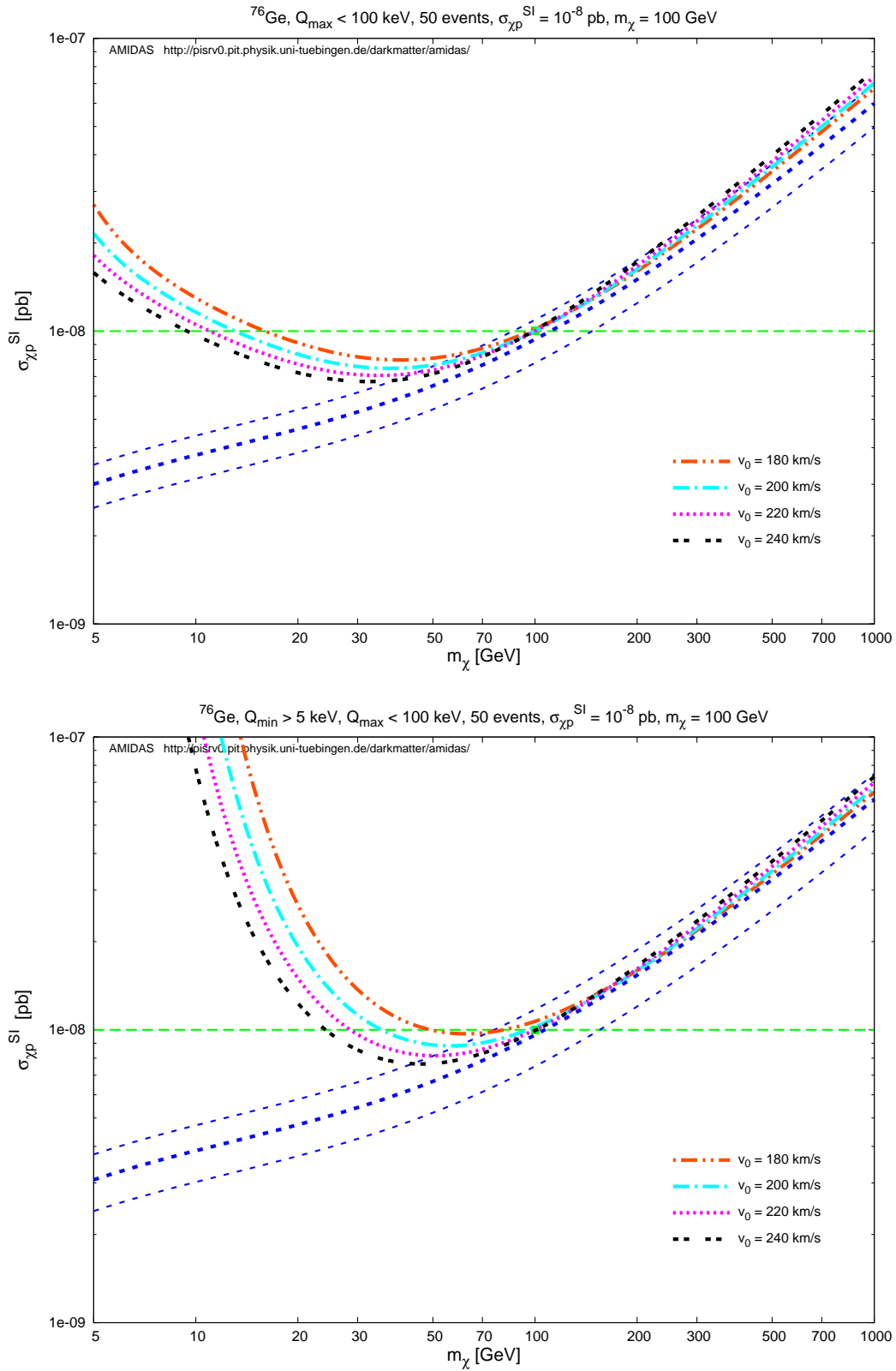


Figure 3: The SI WIMP–proton cross section $\sigma_{\chi p}^{\text{SI}}$ estimated by Eqs. (17) and (12) and the lower and upper bounds of its 1σ statistical uncertainty as functions of the WIMP mass (dashed blue curves) for a ^{76}Ge target. The input WIMP mass is 100 GeV. The threshold energies have been set as 0 (upper) and 5 (lower) keV, respectively. The four extra curves have been drawn conventionally by using the shifted Maxwellian velocity distribution with four Sun’s orbital velocities: $v_0 = 180$ km/s (dash–double–dotted orange), $v_0 = 200$ km/s (dash–dotted cyan), $v_0 = 220$ km/s (dotted magenta), $v_0 = 240$ km/s (double–dotted black), and the form factor given in Eq. (20). The other parameters are as in Fig. 2. See the text for further details.

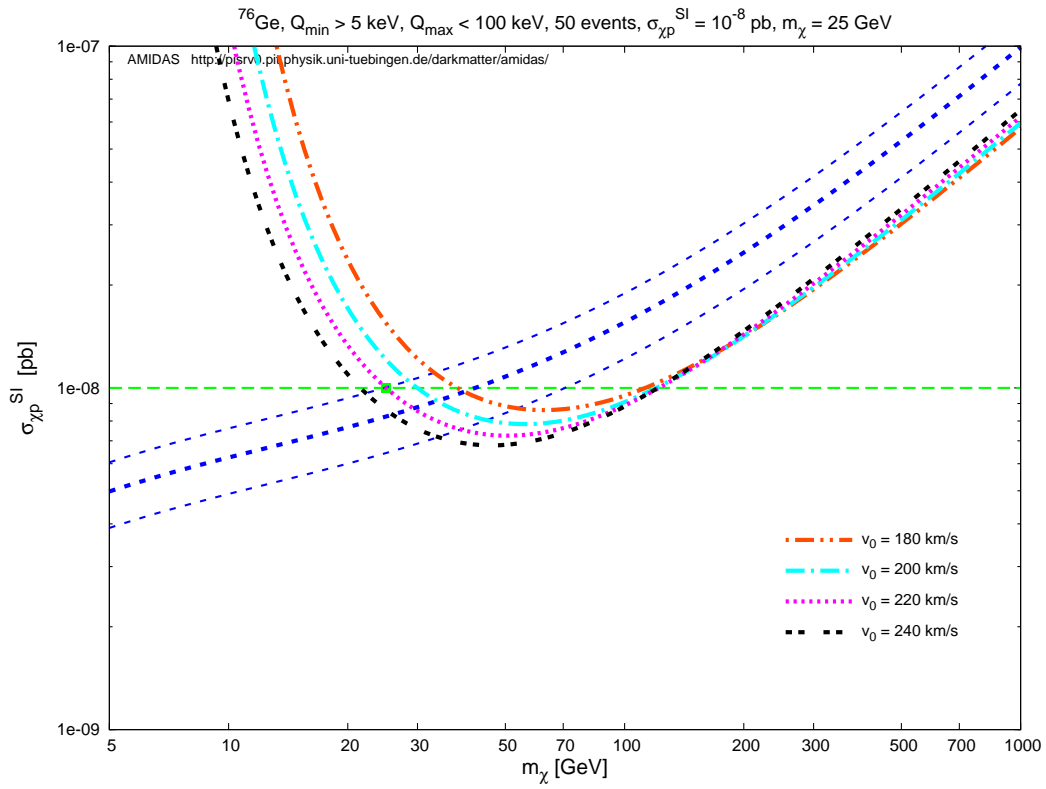
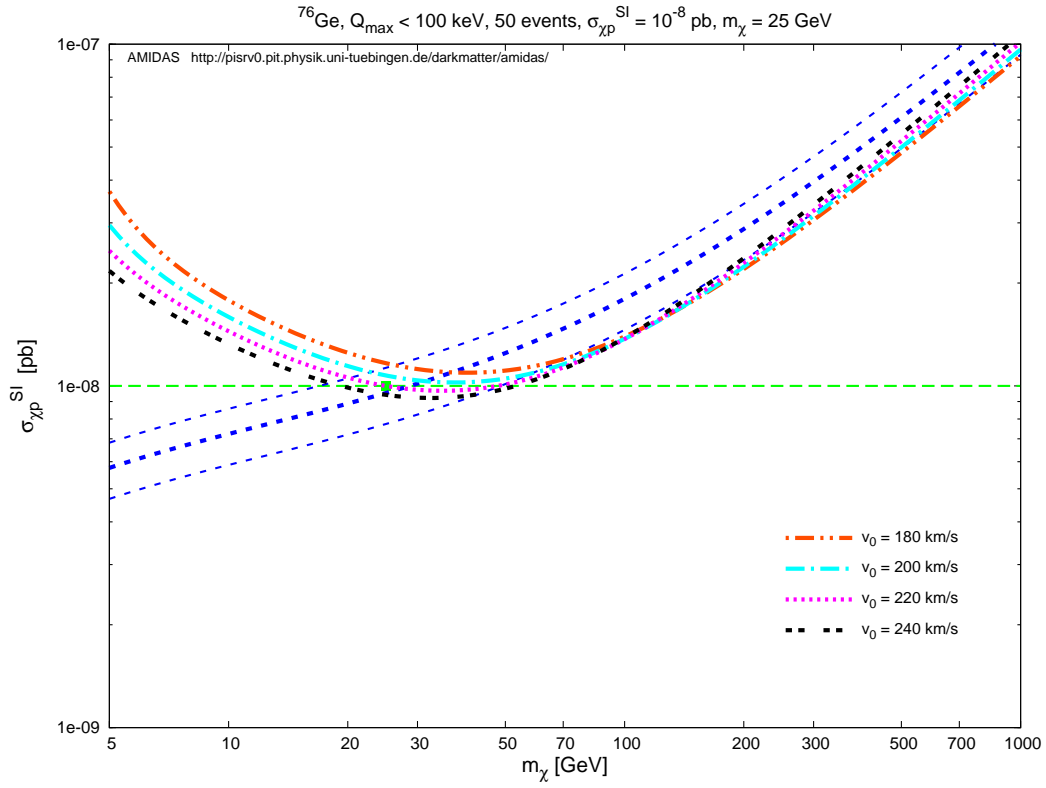


Figure 4: As in Figs. 3, except that the input WIMP mass is only 25 GeV here.

($20 \text{ GeV} \lesssim m_\chi \lesssim 50 \text{ GeV}$ and $7 \times 10^{-9} \text{ pb} \lesssim \sigma_{\chi\text{p}}^{\text{SI}} \lesssim 1.1 \times 10^{-8} \text{ pb}$ in our simulation) from a single experiment. However, remind that with a fixed maximal cut-off energy and a number of total events, the higher the threshold energy, the larger the required exposure. Moreover, as we can see in the lower frame of Figs. 4, the prefactor, or equivalently, I_0 , in Eq. (17) is *underestimated* due to a low (in contrast to the case shown in Figs. 1 and 2) kinematic maximum of the recoil energy. For a WIMP mass of 25 GeV, this kinematic maximum for a Ge target is 52.6 keV. Remind also that the recoil energy spectrum is approximately exponential, thus between $Q = 0$ and $Q = Q_{\text{max,kin}} = 52.6 \text{ keV}$, only $\sim 53\%$ of the total events are with energies $Q \geq Q_{\text{min}} = 5 \text{ keV}$. Due to this underestimate of I_0 , we will see later that the non-negligible threshold energy could cause serious problem once the WIMPs are (pretty) light.

3 Estimating the SI WIMP–nucleon coupling

In this section I consider further the case that two (or more) experiments with different target nuclei observe positive WIMP signals.

3.1 Combining different experiments

In Figs. 5 I show the SI WIMP–proton cross sections $\sigma_{\chi\text{p}}^{\text{SI}}(m_\chi)$ estimated by Eqs. (17) and (12) as functions of the WIMP mass on the $\sigma_{\chi\text{p}}^{\text{SI}} - m_\chi$ plane for four different target nuclei: ^{76}Ge (dashed blue), ^{28}Si (dotted magenta), ^{40}Ar (dash-double-dotted orange), and ^{136}Xe (long-dash-dotted cyan). Not surprisingly, all four curves pass through (approximately) the same values of m_χ and $\sigma_{\chi\text{p}}^{\text{SI}}$. It is in fact one of the basic ideas of the model-independent determination of the WIMP mass [6, 7] mentioned in the introduction⁸. However, one can also find here that the (approximately) common values of m_χ and $\sigma_{\chi\text{p}}^{\text{SI}}$ are somehow *underestimated*, especially for the heavier input WIMP mass (see the lower frame). In Ref. [7], we discussed this phenomenon and introduced therefore an algorithmic procedure to correct this systematic deviation by matching the maximal cut-off energies of different targets. In Figs. 5 the vertical double-dashed black lines show the reconstructed WIMP masses and the lower and upper bounds of their 1σ statistical uncertainties estimated by this algorithmic procedure.

Once the WIMP mass m_χ on the right-hand side of Eq. (17) can be determined by means of the model-independent method with two different target nuclei, one can estimate the SI WIMP–proton coupling (cross section) straightforwardly. Here $r(Q_{\text{min}})$ and I_0 in the prefactor can be estimated from either one of the data sets used for determining m_χ or a third (independent) experiment. In Figs. 6, I show the reconstructed spin-independent WIMP–proton coupling $|f_{\text{p}}|_{\text{rec}}^2$ as a function of the *input* WIMP mass $m_{\chi,\text{in}}$. The expected number of total events in each data set has been set as 50 events on average under the experimental maximal cut-off energy Q_{max} set as 100 GeV for all targets; the experimental threshold energies are assumed to be negligible. Four nuclei: ^{76}Ge , ^{28}Si , ^{40}Ar , and ^{136}Xe have been chosen for estimating $r(Q_{\text{min}})$ and I_0 in Eq. (17). Following our work on the determination of the WIMP mass [7], ^{28}Si and ^{76}Ge have been chosen as two target nuclei for estimating m_χ in Eq. (17).

As a comparison to the use of the reconstructed WIMP mass (solid red), we consider here also the case that the WIMP mass m_χ in Eq. (17) can be determined from some other (collider) experiments (dashed blue) with a higher precision. The input (true) WIMP mass has been used with an overall uncertainty of 5% for this case. Note that, firstly, in order to avoid complicated calculations of the correlations between the uncertainty on m_χ estimated by the algorithmic

⁸A brief review of the determination of the WIMP mass is given in the appendix.

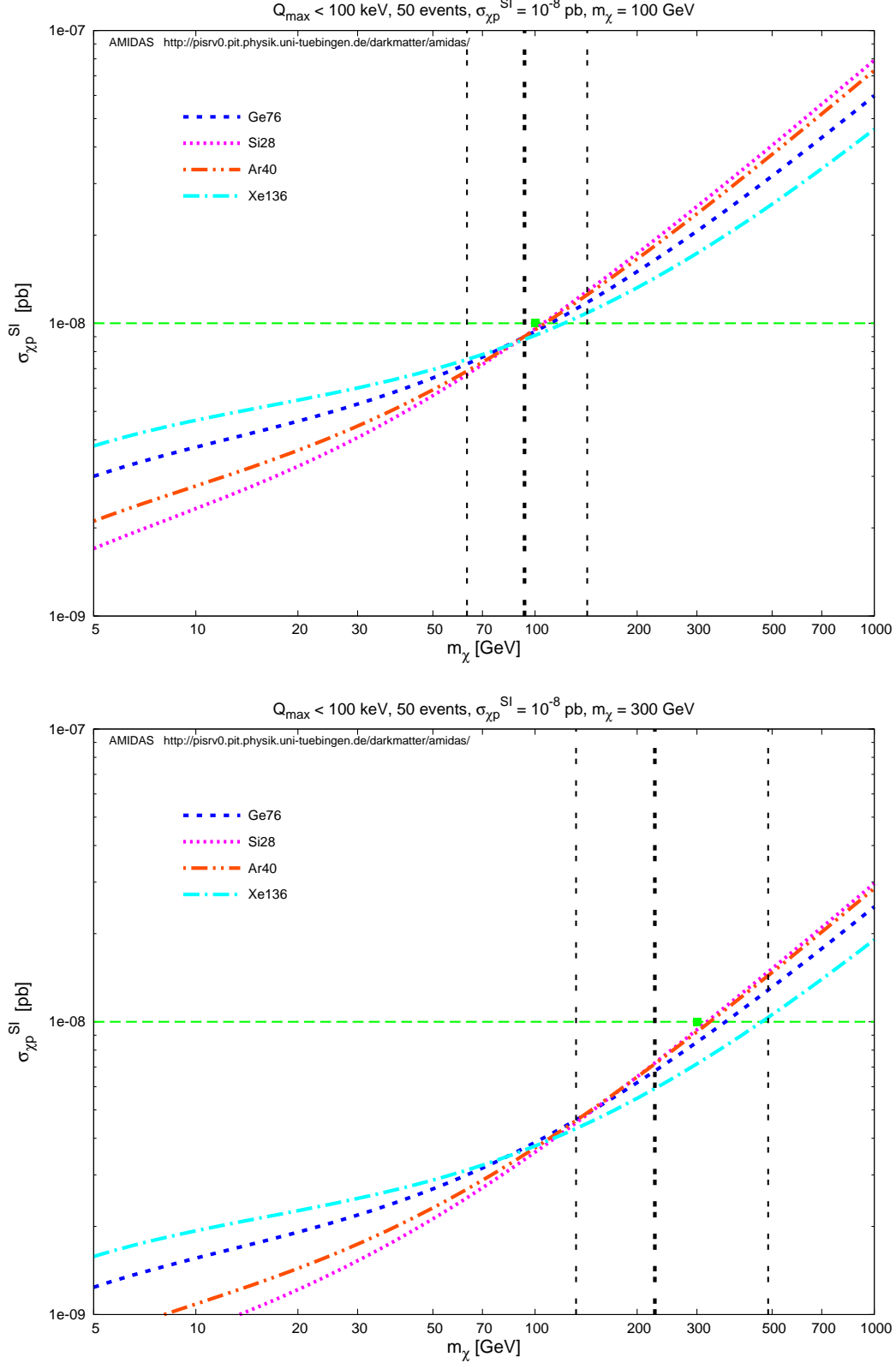


Figure 5: The SI WIMP–proton cross sections $\sigma_{\chi p}^{\text{SI}}(m_\chi)$ estimated by Eqs. (17) and (12) as functions of the WIMP mass for four different target nuclei: ^{76}Ge (dashed blue), ^{28}Si (dotted magenta), ^{40}Ar (dash–double–dotted orange), and ^{136}Xe (long–dash–dotted cyan). The input WIMP mass has been set as 100 (upper) and 300 (lower) GeV. The threshold energies for all targets are assumed to be negligible. The vertical double–dashed black lines show the reconstructed WIMP masses and the lower and upper bounds of their 1σ statistical uncertainties estimated by the algorithmic procedure introduced in Ref. [7]. The other parameters are as in Fig. 2.

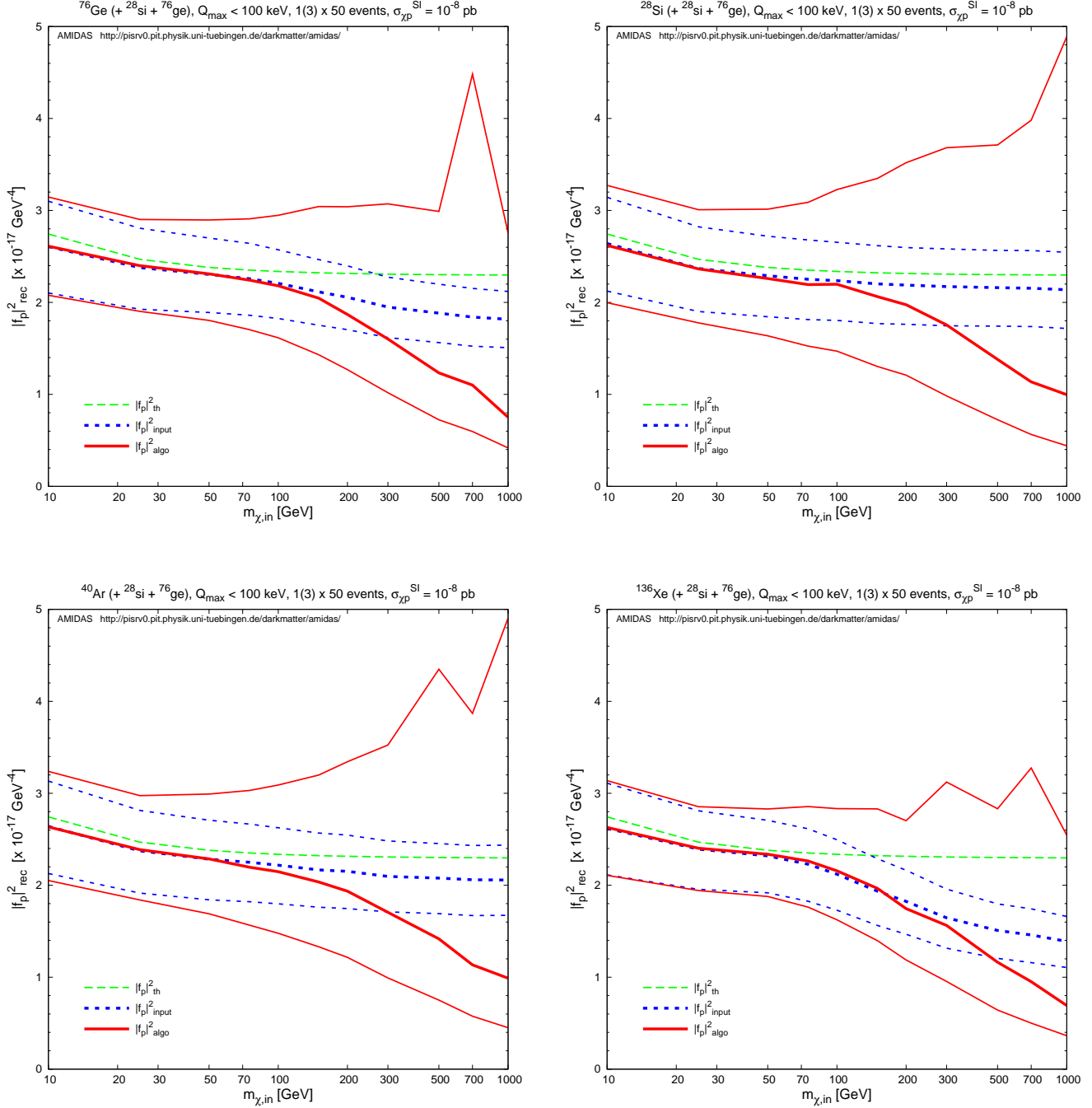


Figure 6: The reconstructed SI WIMP–proton couplings $|f_p|^2_{\text{rec}}$ and the lower and upper bounds of their 1σ statistical uncertainties as functions of the *input* WIMP mass $m_{\chi,\text{in}}$. The long-dashed green curve indicates the theoretical value of the SI coupling. The solid red and dashed blue curves indicate the reconstructed SI couplings estimated with the reconstructed and the input (with an overall uncertainty of 5%) WIMP masses. ^{76}Ge , ^{28}Si , ^{40}Ar , and ^{136}Xe four nuclei have been chosen for estimating $r(Q_{\min})$ and I_0 in Eq. (17). Following our work in Ref. [7], ^{28}Si and ^{76}Ge (labeled in the plots with small letters for their chemical symbols to indicate the independence of these data sets of the data set of the first nucleus) have been chosen as two target nuclei for reconstructing the WIMP mass m_χ . Parameters are as in Figs. 5. Note that for $m_{\chi,\text{in}} \geq 500$ GeV the upper bounds of the statistical uncertainty are systematically *underestimated*. See the text for further details.

procedure and those on $r(Q_{\min})$ and I_0 , we have assumed here that the two data sets with the Ge (Si) nucleus are independent of each other⁹. Secondly, an upper cut-off limit on the reconstructed WIMP mass has been set as 3000 GeV in our simulation. But, due to the very few number of events, the *upper* bounds of the 1σ statistical uncertainty on the reconstructed mass for heavier input masses exceed this limit. Hence, in the heavy mass range ($m_{\chi,\text{in}} \geq 500$ GeV) in Figs. 6 (and also in Figs. 7) the upper bounds of the 1σ statistical uncertainty on the reconstructed SI couplings with all four targets are *systematically underestimated*.

It can however be found in Figs. 6 that, firstly, the reconstructed coupling $|f_p|_{\text{rec}}^2$ estimated with the input (true) WIMP mass (dashed blue curves) for all four targets are *underestimated* for WIMP masses $m_\chi \gtrsim 100$ GeV; for the heavier target nuclei, Ge and Xe, this deviation is larger than for the lighter nuclei, Si and Ar. This is caused by the underestimate of I_0 in Eq. (17), which we found in Figs. 1 and 2 and discussed there. Secondly, due to an *underestimate* of the reconstructed WIMP mass¹⁰, the reconstructed couplings $|f_p|_{\text{rec}}^2$ with the reconstructed WIMP mass (solid red curves) for all four targets are more strongly underestimated than those with the true WIMP mass for WIMP masses $m_\chi \gtrsim 100$ GeV. Moreover, for lighter WIMP masses ($m_{\chi,\text{in}} \lesssim 100$ GeV), the reconstructed $|f_p|_{\text{rec}}^2$ for all targets are also a bit *underestimated*. This could possibly be caused by the statistical fluctuation due to the pretty few events.

The systematic deviation of the reconstructed couplings with both reconstructed and real WIMP mass caused by the underestimate of I_0 reflects the fact that the heavier the target nucleus, the more the contribution from WIMPs with higher velocities to the recoil spectrum. This observation implies that lighter nuclei could be better for estimating I_0 , and in turn for reconstructing the SI WIMP–nucleon coupling. This can be seen more clearly from the difference of the reconstructed $|f_p|^2$ with Si, Ar and Ge, Xe for the case with the input (true) WIMP mass (blue dashed curves). However, Figs. 6 show also that the statistical uncertainties on $|f_p|^2$ estimated with the lighter nuclei are a bit larger than those with the heavier nuclei. And for heavier WIMP masses the deviation of I_0 could in principle be alleviated by extending the experimental maximal cut-off energy Q_{\max} to higher energy ranges, as discussed in the previous section. Moreover, remind that we simulated here with the same expected event number for all four target nuclei. In practice, we could measure (much) less WIMP events in experiments with lighter target nuclei ($dR/dQ \propto A^2$). This indicates also a larger statistical uncertainty.

Nevertheless, our simulations shown in Figs. 6 demonstrate that, firstly, in spite of the systematic deviation for heavier WIMP masses due to the underestimate of I_0 , the true value of $|f_p|^2$ always lies within the 1σ statistical uncertainty intervals. Secondly, for a WIMP mass of 100 GeV, one could in principle estimate the squared SI WIMP–proton coupling with a statistical uncertainty of $\sim 40\%$ for the Si and Ar targets or of only $\sim 30\%$ for the Ge and Xe targets with only 50 events from one experiment¹¹. This is much smaller than the uncertainty on the estimate of the local Dark Matter density (of a factor of 2 or even larger).

⁹The formulae needed for calculating the correlations between the uncertainties on the prefactor and on the WIMP mass estimated by two basic expressions given in Eqs. (31) and (A21) (*not* by the algorithmic procedure) are given in the appendix.

¹⁰The WIMP mass has been reconstructed by a different program than that used in Ref. [7].

¹¹Note that these uncertainties have been estimated by

$$\frac{\left| 1\sigma \text{ upper/lower bound of } |f_p|^2 - |f_p|_{\text{rec}}^2 \right|}{|f_p|_{\text{rec}}^2}, \quad (22)$$

and, as shown in Figs. 6 as well as in Figs. 7, are asymmetric since the upper/lower uncertainties on the reconstructed WIMP mass are asymmetric [7].

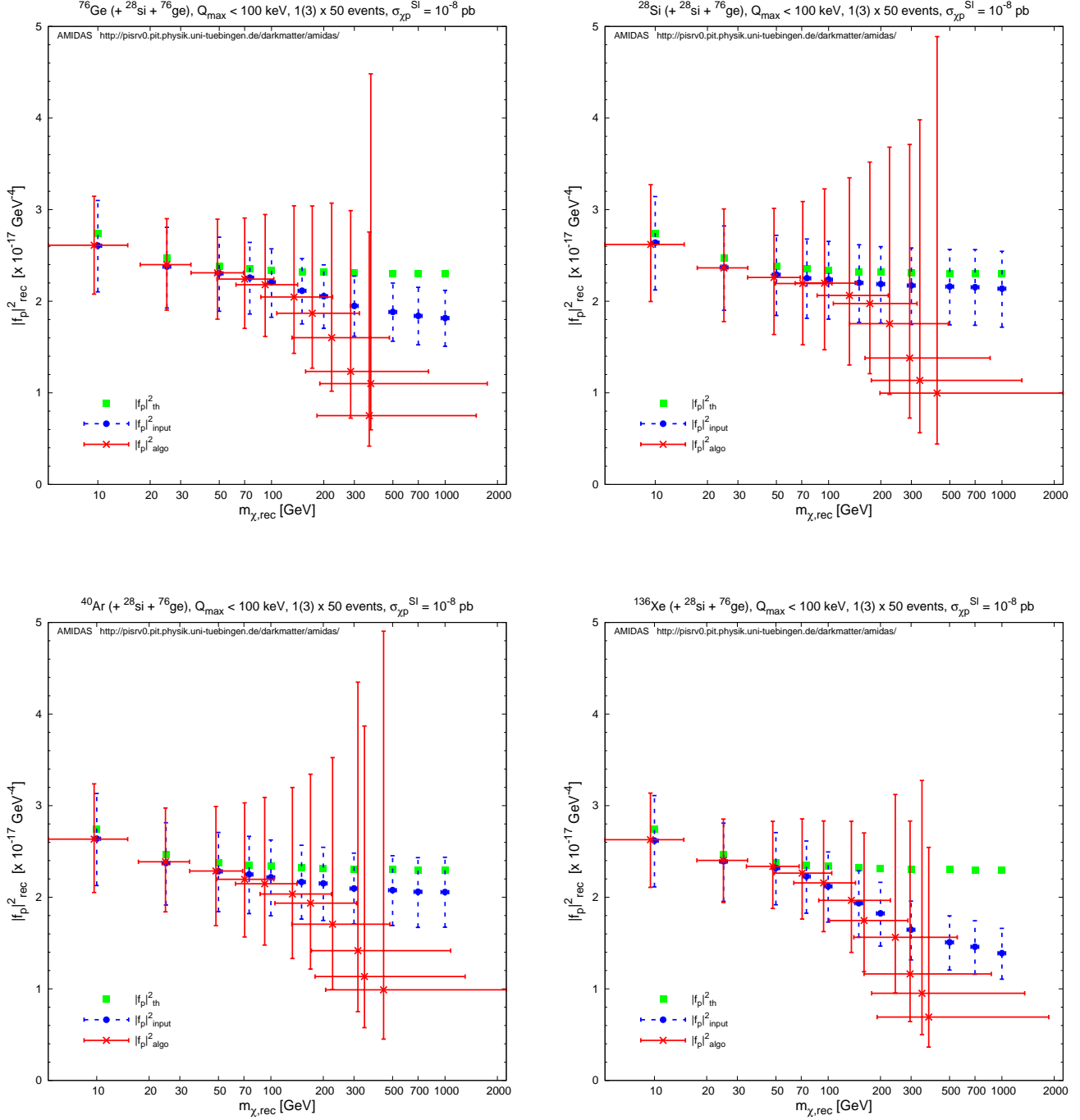


Figure 7: The reconstructed SI WIMP-proton couplings $|f_p|^2_{\text{rec}}$ and the *reconstructed* WIMP mass $m_{\chi, \text{rec}}$ estimated by the method described in Ref. [7] with the Si and Ge targets on the cross section (coupling) versus WIMP mass plane. The filled green squares indicate the input WIMP masses and the theoretical values of the SI coupling. The red crosses (filled blue circles) indicate the reconstructed (input) WIMP masses and the reconstructed SI couplings estimated with these WIMP masses. The horizontal (vertical) solid red and dashed blue lines show the 1σ statistical uncertainties on $m_{\chi, \text{rec}}$ ($|f_p|^2_{\text{rec}}$). Parameters are as in Fig. 6.

3.2 Constraints on the cross section–mass plane

Since by using two (or three) data sets the squared SI WIMP–nucleon coupling $|f_p|^2$ can be estimated from experimental data directly without knowing the true value of the WIMP mass m_χ , and the same data sets can also be used to reconstruct m_χ , one can practically combine the reconstructed $|f_p|^2$ with the reconstructed m_χ together on the cross section–mass plane. In Figs. 7 I show the reconstructed SI coupling $|f_p|_{\text{rec}}^2$ and the *reconstructed* WIMP mass $m_{\chi,\text{rec}}$ estimated by the algorithmic procedure described in Ref. [7] with the Si and Ge targets on the cross section (coupling) versus WIMP mass plane. The horizontal (vertical) solid red and long-dashed blue lines show the 1σ statistical uncertainties on $m_{\chi,\text{rec}}$ ($|f_p|_{\text{rec}}^2$). It can be seen that the 1σ statistical uncertainty areas of the reconstructed WIMP mass and its coupling can always cover their true values up to an input mass of ~ 1 TeV, although both of them are underestimated.

The emphasis here is that, while by the conventional analyses for determining the WIMP mass and its SI coupling on nucleons (see e.g., [21, 22, 23]) one needs a model of the velocity distribution of halo WIMPs, one can estimate m_χ and $|f_p|^2$ *separately* by the method presented here with *neither* prior knowledge of each other *nor* an assumption about the WIMP velocity distribution. Certainly, how well one can estimate these two quantities depends not only on the event number but also on the target nucleus, as discussed in Ref. [7] and shown in Figs. 6 and 7.

In Figs. 3 and 4 we saw that the non-negligible threshold energy could allow us to give more strict constraints on the WIMP mass and its SI coupling on nucleons. In Figs. 8 we therefore take into account a minimal cut-off energy $Q_{\min} = 5$ keV for the *first* Ge target used for estimating $r(Q_{\min})$ and I_0 . It can be seen obviously that, for lighter WIMP masses ($m_\chi \lesssim 50$ GeV) the reconstructed coupling $|f_p|_{\text{rec}}^2$ is (strongly) *underestimated* for both cases with the reconstructed and the input (true) WIMP masses. As discussed at the end of the previous section, this is caused by a (very) low kinematic maximum of the recoil energy and, consequently, the underestimate of I_0 . For a WIMP mass of 10 GeV, this kinematic maximum is just 11.8 keV and between $Q = 0$ and $Q = Q_{\max,\text{kin}} = 11.8$ keV, only $\sim 6.4\%$ of the total events are with energies $Q \geq Q_{\min} = 5$ keV! In contrast, for heavier WIMP masses ($m_\chi \gtrsim 50$ GeV), the non-negligible threshold energy causes only slightly larger statistical uncertainties on the reconstructed SI couplings.

So far we have assumed that each experiment “only” has an exposure corresponding to 50 total events. In Figs. 9 we raise this number by a factor of 10. Not surprisingly, all uncertainties on both the reconstructed WIMP mass and the reconstructed SI couplings shrink by a factor $\gtrsim 3$ compared to the results shown in Figs. 7. Moreover, the small underestimate for lighter WIMP masses ($m_\chi \lesssim 50$ GeV) found in our simulations with only 50 events (see Figs. 6) disappears now. Note that, for heavier WIMP masses ($m_\chi \gtrsim 500$ GeV), the upper bounds of the 1σ statistical uncertainty on the reconstructed WIMP masses is now down to below our cut-off limit.

4 Estimating the SD WIMP–nucleon couplings

For the sake of completeness, I consider briefly in this section the case that the spin-dependent WIMP–nucleus interaction dominates over the spin-independent one. Then the WIMP–nucleus cross section σ_0 in Eq. (2) can be expressed as [1, 2]:

$$\sigma_0^{\text{SD}} = \left(\frac{32}{\pi}\right) G_F^2 m_{\text{r},\text{N}}^2 \left(\frac{J+1}{J}\right) \left[\langle S_p \rangle a_p + \langle S_n \rangle a_n\right]^2. \quad (23)$$

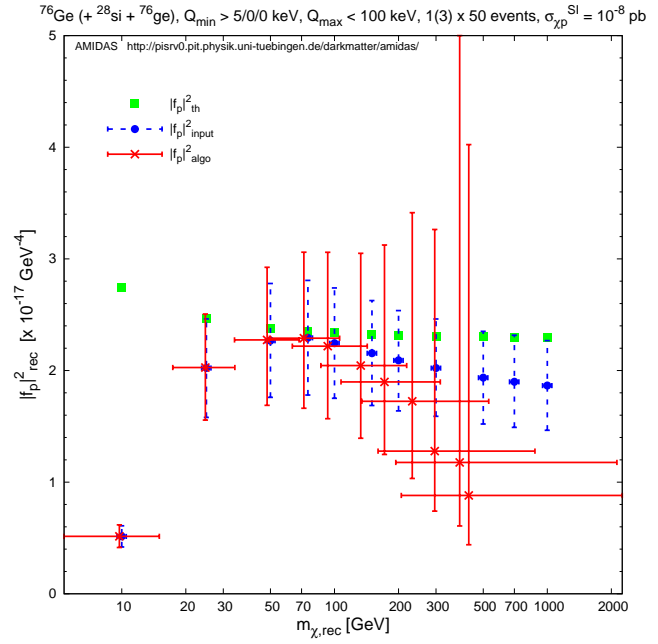
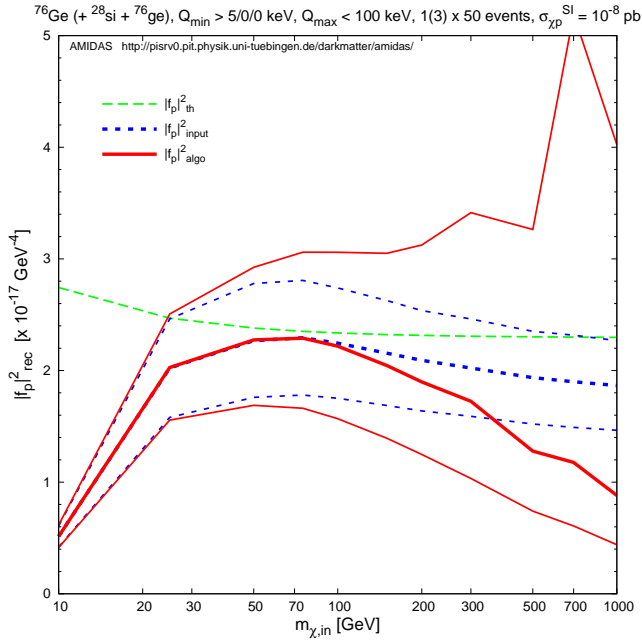


Figure 8: As in Figs. 6 and 7, except that the experimental minimal cut-off energy for the *first* Ge target has been set as 5 keV.

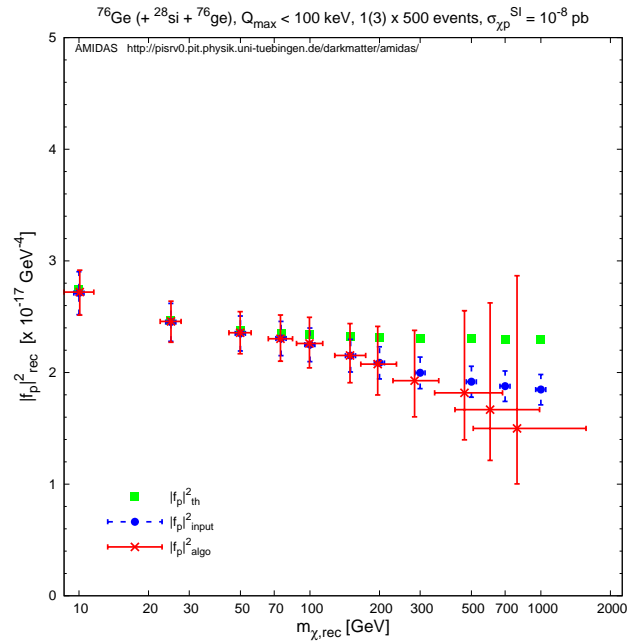
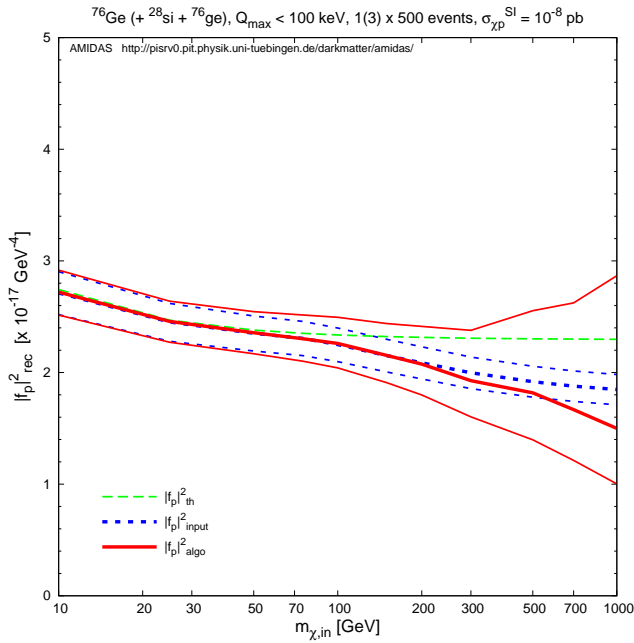


Figure 9: As in Figs. 6 and 7, except that the expected number of total events in *all three* experiments has been set as 500.

Here G_F is the Fermi constant, J is the total spin of the target nucleus, $\langle S_{(p,n)} \rangle$ are the expectation values of the proton and neutron group spins, and $a_{(p,n)}$ are the effective SD WIMP couplings on protons and on neutrons. For the SD WIMP–nucleus cross section, it is usually assumed that only unpaired nucleons contribute significantly to the total cross section, as the spins of the nucleons in a nucleus are systematically anti-aligned¹². Under this assumption, the SD WIMP–nucleus cross section given above can be reduced to

$$\begin{aligned}\sigma_0^{\text{SD}} &= \left(\frac{32}{\pi}\right) G_F^2 m_{r,N}^2 \left(\frac{J+1}{J}\right) \langle S_{(p,n)} \rangle^2 |a_{(p,n)}|^2 \\ &= \frac{4}{3} \left(\frac{J+1}{J}\right) \langle S_{(p,n)} \rangle^2 \left(\frac{m_{r,N}}{m_{r,(p,n)}}\right)^2 \sigma_{\chi(p,n)}^{\text{SD}}.\end{aligned}\quad (24)$$

Since for a proton or a neutron $J = \frac{1}{2}$ and $\langle S_p \rangle$ or $\langle S_n \rangle = \frac{1}{2}$, the SD WIMP cross section on protons or on neutrons can be given as

$$\sigma_{\chi(p,n)}^{\text{SD}} = \left(\frac{24}{\pi}\right) G_F^2 m_{r,(p,n)}^2 |a_{(p,n)}|^2. \quad (25)$$

By comparing Eq. (24) with the second expression in Eq. (11), the *squared* SD WIMP couplings on protons and on neutrons can be obtained from Eq. (17) straightforwardly as

$$|a_{(p,n)}|^2 = \frac{1}{\rho_0} \left[\frac{\pi}{32\sqrt{2}} \left(\frac{J}{J+1}\right) \left(\frac{1}{\mathcal{E} G_F^2 \langle S_{(p,n)} \rangle^2 \sqrt{m_N}}\right) \right] \left[\frac{2Q_{\min}^{1/2} r(Q_{\min})}{F_{\text{SD}}^2(Q_{\min})} + I_0 \right] (m_\chi + m_N). \quad (26)$$

Note that, for estimating I_0 here by using Eq. (15), the elastic nuclear form factor $F^2(Q)$ must be chosen for the SD interaction.

As the use of Eq. (17) for constraining the SI coupling $|f_p|^2$ discussed in Sec. 2, by assuming that the SD WIMP–nucleus interaction contributes to the total cross section σ_0 dominantly, and combining with the conventional cross section–WIMP mass analysis, one could in principle use Eq. (26) to give the *lower* bounds of the WIMP mass and its SD couplings on protons and on neutrons from a *single* experiment with a target nucleus having spin sensitivity (almost) only on protons or on neutrons. By comparing these results with those obtained from the SI case, one could examine backwards the assumption for a dominant SD WIMP interaction. Meanwhile, as discussed in Sec. 3, once the WIMP mass m_χ can be determined, one could then estimate the SD WIMP–nucleon cross sections by Eqs. (26) and (25) straightforwardly.

Furthermore, by combining two target nuclei, one (X) of them has (almost) only spin sensitivity on protons and the other one (Y) on neutrons, one can easily find an expression for the *ratio* between two SD WIMP–nucleon couplings from Eq. (26) as

$$\begin{aligned}\frac{a_n}{a_p} &= \pm \left[\left(\frac{J_Y}{J_Y+1}\right) \left(\frac{\mathcal{R}_{\sigma,Y}}{\langle S_n \rangle_Y^2}\right) \left(\frac{m_\chi + m_Y}{\sqrt{m_Y}}\right) \right]^{1/2} \left[\left(\frac{J_X+1}{J_X}\right) \left(\frac{\langle S_p \rangle_X^2}{\mathcal{R}_{\sigma,X}}\right) \left(\frac{\sqrt{m_X}}{m_\chi + m_X}\right) \right]^{1/2} \\ &= \pm \frac{\mathcal{R}_{J,n,Y}}{\langle S_n \rangle_Y} \cdot \frac{\langle S_p \rangle_X}{\mathcal{R}_{J,n,X}}.\end{aligned}\quad (27)$$

Here I have used [23]

$$\mathcal{R}_{\sigma,X} \equiv \frac{1}{\mathcal{E}_X} \left[\frac{2Q_{\min,X}^{1/2} r_X(Q_{\min,X})}{F_X^2(Q_{\min,X})} + I_{0,X} \right], \quad (28)$$

¹²However, more detailed nuclear spin structure calculations show that the even group of nucleons has sometimes also non-negligible spin [1].

and defined

$$\mathcal{R}_{J,n,X} \equiv \left[\left(\frac{J_X}{J_X + 1} \right) \frac{\mathcal{R}_{\sigma,X}}{\mathcal{R}_{n,X}} \right]^{1/2} \quad (29)$$

for $n \neq 0$, with

$$\mathcal{R}_{n,X} \equiv \left[\frac{2Q_{\min,X}^{(n+1)/2} r_X(Q_{\min,X}) / F_X^2(Q_{\min,X}) + (n+1)I_{n,X}}{2Q_{\min,X}^{1/2} r_X(Q_{\min,X}) / F_X^2(Q_{\min,X}) + I_{0,X}} \right]^{1/n}; \quad (30)$$

$\mathcal{R}_{\sigma,Y}$, $\mathcal{R}_{J,n,Y}$, and $\mathcal{R}_{n,Y}$ can be defined analogously. Here $m_{(X,Y)}$ and $F_{(X,Y)}(Q)$ are the masses and the form factors of the nucleus X and Y , respectively, $r_{(X,Y)}(Q_{\min,(X,Y)})$ refer to the counting rates for the target X and Y at the respective lowest recoil energies included in the analysis, and $\mathcal{E}_{(X,Y)}$ are the experimental exposures with the target X and Y . For the cancellation of the factors involving m_χ in the first line of Eq. (27), I used the general estimator for the WIMP mass given in Refs. [6, 7]:

$$m_\chi|_{\langle v^n \rangle} = \frac{\sqrt{m_X m_Y} - m_X (\mathcal{R}_{n,X} / \mathcal{R}_{n,Y})}{\mathcal{R}_{n,X} / \mathcal{R}_{n,Y} - \sqrt{m_X / m_Y}}. \quad (31)$$

Note that Eq. (27) derived here is in fact a special case of the general expression (16) given in Refs. [18, 17]. Detailed discussions about model-independent determinations of the ratios between different WIMP–nucleon couplings/cross sections can be found in Refs. [18, 17, 19].

5 Summary and conclusions

In this paper I presented the method for estimating the spin-independent WIMP–nucleon coupling from elastic WIMP–nucleus scattering experiments. This method is independent of the velocity distribution of halo WIMPs as well as (practically) of the as yet unknown WIMP mass. Assuming that an exponential-like shape of the recoil spectrum is confirmed from experimental data, the required information are only the measured recoil energies and the number of events in the first energy bin from at least two experiments with different target nuclei as well as the unique assumption for the local WIMP density.

In Sec. 2 I rederived the expression for estimating the (squared) SI WIMP–nucleon coupling $|f_p|^2$ as a function of the (unknown) WIMP mass [17]. Then I demonstrated that, by comparing the constrained area estimated by this method to that given by the conventional analysis with an assumed halo model, one could in principle – for the first step with only one experiment observing positive signals – give the lower bounds of the WIMP mass and its SI cross section on nucleons from a single experiment.

For the next step, I discussed in Sec. 3 that, by using measured recoil energies from two (or three) experiments with different target nuclei, we could not only determine the WIMP mass as discussed in Refs. [6, 7], but also estimate the SI WIMP–nucleon coupling, with neither prior knowledge of each other nor an assumption for the velocity distribution of halo WIMPs.

However, due to the degeneracy between the local WIMP density and the WIMP–nucleus cross section, it is impossible to determine both of them independently. As the simplest way one has thus to make an assumption for the local WIMP density. Nevertheless, since the SI WIMP–nucleon coupling is inversely proportional to the local WIMP density, whose common value would possibly be underestimated, one can then at least give an upper bound on this coupling. Moreover, our simulations show that, in spite of the very few ($\mathcal{O}(50)$) total events

from one experiment, for a WIMP mass of 100 GeV, the SI WIMP–nucleon coupling can be estimated with a statistical uncertainty of only $\sim 15\%$; it leads to an uncertainty on the SI WIMP–nucleon cross section of only $\sim 30\%$, which is (much) smaller than the uncertainty on the estimate of the local Dark Matter density (of a factor of 2 or even larger).

Our simulations show also that, due to (mainly) the experimental maximal cut-off energy, the SI WIMP coupling could be underestimated for heavier WIMP masses, especially with heavy target nuclei, e.g., Ge or Xe. However, since the kinematic maximum of recoil energies for heavier WIMP masses and/or with heavy target nuclei are (much) higher than for lighter WIMP masses with light nuclei, one could practically alleviate this systematic deviation by extending the detector sensitivity to higher energy ranges. Moreover, due to the fairly large statistical uncertainty, the true value of the SI WIMP–nucleon coupling lies always within the 1σ statistical uncertainty interval.

In Sec. 4 I turned to consider the case that the spin-dependent WIMP–nucleus interaction dominates over the SI one. By assuming (naively) that only unpaired nucleons contribute significantly to the total WIMP–nucleus cross section, I gave also the expression for estimating the (squared) SD WIMP–nucleon couplings $|a_{(p,n)}|^2$ as functions of the (unknown) WIMP mass. As for the SI case, by comparing the constraints estimated by this method to those given by the conventional analysis, we could in principle also give the lower bounds of the WIMP mass and its SD cross sections on nucleons from a single experiment.

Our simulations presented here are based on several simplified assumptions. Firstly, the sample to be analyzed contains only signal events, i.e., is free of background^{13, 14}. Secondly, all experimental systematic uncertainties as well as the uncertainty on the measurement of the recoil energy have been ignored. The energy resolution of most currently running and projected detectors is so good that its uncertainty can be neglected compared to the statistical uncertainty with (very) few events in the foreseeable future.

A non-negligible threshold energy makes the conventional model-dependent analysis less sensitive on light WIMPs ($m_\chi \lesssim 20$ GeV), it could however give us more strict constraints on the lower bounds of the WIMP mass and its couplings on nucleons. In contrast, our simulation shows that, by using our model-independent method, the non-negligible threshold energy could cause not only a larger statistical uncertainty on the reconstructed couplings, but also a significant underestimate if WIMPs are (very) light.

In summary, I demonstrated in this paper the use of our new method for estimating the spin-independent WIMP–nucleon coupling with neither a prior knowledge of the WIMP mass nor an assumption for the velocity distribution of halo WIMPs. By combining with information on the ratios between different WIMP–nucleon couplings/cross sections, which could also be determined model-independently [18, 17, 19], one could in principle also estimate the absolute values of the spin-dependent cross sections. This information combined with the reconstructed WIMP mass will allow us not only to constrain the parameter space in different extensions of the Standard Model of particle physics [30, 31, 16], but also to identify WIMPs among new particles produced at colliders [8]. Furthermore, knowledge of the WIMP mass and its couplings could not only offer a new approach for estimating the local WIMP density, but also permit the prediction of the WIMP annihilation cross section and the event rate in indirect Dark Matter detection experiments [1, 2].

¹³For background discrimination techniques and status in currently running and projected direct detection experiments see e.g., [24, 25, 26, 27].

¹⁴For detailed simulations and discussions about effects of residue background events on the reconstructions of the WIMP mass and its SI coupling on nucleons see [28, 29].

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A Formulae needed in Secs. 2 and 3

Here I list all formulae needed for our model-independent data analyses described in this article. Detailed derivations and discussions can be found in Refs. [5, 7].

A.1 Estimating $r(Q_{\min})$ and $I_n(Q_{\min}, Q_{\max})$

First, consider experimental data described by

$$Q_n - \frac{b_n}{2} \leq Q_{n,i} \leq Q_n + \frac{b_n}{2}, \quad i = 1, 2, \dots, N_n, \quad n = 1, 2, \dots, B. \quad (\text{A1})$$

Here the total energy range between Q_{\min} and Q_{\max} has been divided into B bins with central points Q_n and widths b_n . In each bin, N_n events will be recorded. Since the recoil spectrum dR/dQ is expected to be approximately exponential, the following ansatz for the *measured* recoil spectrum (*before* normalized by the experimental exposure \mathcal{E}) in the n th bin has been introduced [5]:

$$\left(\frac{dR}{dQ}\right)_{\text{expt}, n} \equiv \left(\frac{dR}{dQ}\right)_{\text{expt}, Q \simeq Q_n} \equiv r_n e^{k_n(Q - Q_{s,n})}. \quad (\text{A2})$$

Here r_n is the standard estimator for $(dR/dQ)_{\text{expt}}$ at $Q = Q_n$:

$$r_n = \frac{N_n}{b_n}, \quad (\text{A3})$$

k_n is the logarithmic slope of the recoil spectrum in the n th Q -bin, which can be computed numerically from the average value of the measured recoil energies in this bin:

$$\overline{Q - Q_n}|_n = \left(\frac{b_n}{2}\right) \coth\left(\frac{k_n b_n}{2}\right) - \frac{1}{k_n}, \quad (\text{A4})$$

where

$$\overline{(Q - Q_n)^\lambda}|_n \equiv \frac{1}{N_n} \sum_{i=1}^{N_n} (Q_{n,i} - Q_n)^\lambda. \quad (\text{A5})$$

The error on the logarithmic slope k_n can be estimated from Eq. (A4) directly as

$$\sigma^2(k_n) = k_n^4 \left\{ 1 - \left[\frac{k_n b_n/2}{\sinh(k_n b_n/2)} \right]^2 \right\}^{-2} \sigma^2(\overline{Q - Q_n}|_n), \quad (\text{A6})$$

with

$$\sigma^2(\overline{Q - Q_n}|_n) = \frac{1}{N_n - 1} \left[\overline{(Q - Q_n)^2}|_n - \overline{Q - Q_n}|_n^2 \right]. \quad (\text{A7})$$

$Q_{s,n}$ in the ansatz (A2) is the shifted point at which the leading systematic error due to the ansatz is minimal [5],

$$Q_{s,n} = Q_n + \frac{1}{k_n} \ln \left[\frac{\sinh(k_n b_n/2)}{k_n b_n/2} \right]. \quad (\text{A8})$$

Note that $Q_{s,n}$ differs from the central point of the n th bin, Q_n . From the ansatz (A2), the counting rate at $Q = Q_{\min}$ can be calculated by

$$r(Q_{\min}) = r_1 e^{k_1(Q_{\min} - Q_{s,1})}, \quad (\text{A9})$$

and its statistical error can be expressed as

$$\sigma^2(r(Q_{\min})) = r^2(Q_{\min}) \left\{ \frac{1}{N_1} + \left[\frac{1}{k_1} - \left(\frac{b_1}{2} \right) \left(1 + \coth \left(\frac{b_1 k_1}{2} \right) \right) \right]^2 \sigma^2(k_1) \right\}, \quad (\text{A10})$$

since

$$\sigma^2(r_n) = \frac{N_n}{b_n^2}. \quad (\text{A11})$$

Finally, since all I_n are determined from the same data, they are correlated with

$$\text{cov}(I_n, I_m) = \sum_{a=1}^{N_{\text{tot}}} \frac{Q_a^{(n+m-2)/2}}{F^4(Q_a)}, \quad (\text{A12})$$

where the sum runs over all events with recoil energy between Q_{\min} and Q_{\max} . And the correlation between the errors on $r(Q_{\min})$, which is calculated entirely from the events in the first bin, and on I_n is given by

$$\begin{aligned} & \text{cov}(r(Q_{\min}), I_n) \\ &= r(Q_{\min}) I_n(Q_{\min}, Q_{\min} + b_1) \\ & \quad \times \left\{ \frac{1}{N_1} + \left[\frac{1}{k_1} - \left(\frac{b_1}{2} \right) \left(1 + \coth \left(\frac{b_1 k_1}{2} \right) \right) \right] \right. \\ & \quad \times \left. \left[\frac{I_{n+2}(Q_{\min}, Q_{\min} + b_1)}{I_n(Q_{\min}, Q_{\min} + b_1)} - Q_1 + \frac{1}{k_1} - \left(\frac{b_1}{2} \right) \coth \left(\frac{b_1 k_1}{2} \right) \right] \sigma^2(k_1) \right\}; \end{aligned} \quad (\text{A13})$$

note that the sums I_i here only count in the first bin, which ends at $Q = Q_{\min} + b_1$.

On the other hand, with a functional form of the recoil spectrum (e.g., fitted to experimental data), $(dR/dQ)_{\text{expt}}$, one can use the following integral forms to replace the summations given above. Firstly, the average Q -value in the n th bin defined in Eq. (A5) can be calculated by

$$\overline{(Q - Q_n)^\lambda}|_n = \frac{1}{N_n} \int_{Q_n - b_n/2}^{Q_n + b_n/2} (Q - Q_n)^\lambda \left(\frac{dR}{dQ} \right)_{\text{expt}} dQ. \quad (\text{A14})$$

For $I_n(Q_{\min}, Q_{\max})$ given in Eq. (15), we have

$$I_n(Q_{\min}, Q_{\max}) = \int_{Q_{\min}}^{Q_{\max}} \frac{Q^{(n-1)/2}}{F^2(Q)} \left(\frac{dR}{dQ} \right)_{\text{expt}} dQ, \quad (\text{A15})$$

and similarly for the covariance matrix for I_n in Eq. (A12),

$$\text{cov}(I_n, I_m) = \int_{Q_{\min}}^{Q_{\max}} \frac{Q^{(n+m-2)/2}}{F^4(Q)} \left(\frac{dR}{dQ} \right)_{\text{expt}} dQ. \quad (\text{A16})$$

Remind that $(dR/dQ)_{\text{expt}}$ is the *measured* recoil spectrum *before* normalized by the exposure. Finally, $I_i(Q_{\min}, Q_{\min} + b_1)$ needed in Eq. (A13) can be calculated by

$$I_n(Q_{\min}, Q_{\min} + b_1) = \int_{Q_{\min}}^{Q_{\min} + b_1} \frac{Q^{(n-1)/2}}{F^2(Q)} \left[r_1 e^{k_1(Q - Q_{s,1})} \right] dQ. \quad (\text{A17})$$

Note that, firstly, $r(Q_{\min})$ and $I_n(Q_{\min}, Q_{\min} + b_1)$ should be estimated by Eqs. (A9) and (A17) with r_1 , k_1 and $Q_{s,1}$ estimated by Eqs. (A3), (A4), and (A8) in order to use the other formulae for estimating the (correlations between the) statistical errors without any modification. Secondly, $r(Q_{\min})$ and $I_n(Q_{\min}, Q_{\max})$ estimated from a scattering spectrum fitted to experimental data are usually not model-independent any more. Moreover, for estimating the SD WIMP–nucleon couplings by Eq. (26), the elastic nuclear form factor $F^2(Q)$ in Eqs. (15), (A12), (A15), (A16), and (A17) should be understood to be chosen for the SD interaction.

A.2 Determining the WIMP mass m_χ

By requiring that the values of a given moment of $f_1(v)$ estimated by Eq. (13) from two experiments with different target nuclei, X and Y , agree, m_χ appearing in the prefactor α^n on the right-hand side of Eq. (13) can be solved analytically as [6, 7]:

$$m_\chi|_{\langle v^n \rangle} = \frac{\sqrt{m_X m_Y} - m_X (\mathcal{R}_{n,X} / \mathcal{R}_{n,Y})}{\mathcal{R}_{n,X} / \mathcal{R}_{n,Y} - \sqrt{m_X / m_Y}}, \quad (\text{31})$$

with $\mathcal{R}_{n,(X,Y)}$ given by Eq. (30). Note that the general expression (31) can be used either for spin-independent or for spin-dependent scattering, one only needs to choose different form factors under different assumptions; the form factors needed for estimating $I_{n,(X,Y)}$ by Eq. (15) or (A15) are thus also different.

By using the standard Gaussian error propagation, a lengthy expression for the statistical uncertainty on $m_\chi|_{\langle v^n \rangle}$ can be obtained as

$$\begin{aligned} \sigma(m_\chi)|_{\langle v^n \rangle} &= \frac{\sqrt{m_X / m_Y} |m_X - m_Y| (\mathcal{R}_{n,X} / \mathcal{R}_{n,Y})}{(\mathcal{R}_{n,X} / \mathcal{R}_{n,Y} - \sqrt{m_X / m_Y})^2} \\ &\times \left[\frac{1}{\mathcal{R}_{n,X}^2} \sum_{i,j=1}^3 \left(\frac{\partial \mathcal{R}_{n,X}}{\partial c_{i,X}} \right) \left(\frac{\partial \mathcal{R}_{n,X}}{\partial c_{j,X}} \right) \text{cov}(c_{i,X}, c_{j,X}) + (X \rightarrow Y) \right]^{1/2}. \end{aligned} \quad (\text{A18})$$

Here a short-hand notation for the six quantities on which the estimate of m_χ depends has been introduced:

$$c_{1,X} = I_{n,X}, \quad c_{2,X} = I_{0,X}, \quad c_{3,X} = r_X(Q_{\min,X}); \quad (\text{A19})$$

and similarly for the $c_{i,Y}$. Estimators for $\text{cov}(c_i, c_j)$ have been given in Eqs. (A12) and (A13). Explicit expressions for the derivatives of $\mathcal{R}_{n,X}$ with respect to $c_{i,X}$ are:

$$\frac{\partial \mathcal{R}_{n,X}}{\partial I_{n,X}} = \frac{n+1}{n} \left[\frac{F_X^2(Q_{\min,X})}{2Q_{\min,X}^{(n+1)/2} r_X(Q_{\min,X}) + (n+1)I_{n,X} F_X^2(Q_{\min,X})} \right] \mathcal{R}_{n,X}, \quad (\text{A20a})$$

$$\frac{\partial \mathcal{R}_{n,X}}{\partial I_{0,X}} = -\frac{1}{n} \left[\frac{F_X^2(Q_{\min,X})}{2Q_{\min,X}^{1/2} r_X(Q_{\min,X}) + I_{0,X} F_X^2(Q_{\min,X})} \right] \mathcal{R}_{n,X}, \quad (\text{A20b})$$

and

$$\frac{\partial \mathcal{R}_{n,X}}{\partial r_X(Q_{\min,X})} = \frac{2}{n} \left[\frac{Q_{\min,X}^{(n+1)/2} I_{0,X} - (n+1) Q_{\min,X}^{1/2} I_{n,X}}{2Q_{\min,X}^{(n+1)/2} r_X(Q_{\min,X}) + (n+1) I_{n,X} F_X^2(Q_{\min,X})} \right] \times \left[\frac{F_X^2(Q_{\min,X})}{2Q_{\min,X}^{1/2} r_X(Q_{\min,X}) + I_{0,X} F_X^2(Q_{\min,X})} \right] \mathcal{R}_{n,X}; \quad (\text{A20c})$$

explicit expressions for the derivatives of $\mathcal{R}_{n,Y}$ with respect to $c_{i,Y}$ can be given analogously. Note that, firstly, factors $\mathcal{R}_{n,(X,Y)}$ appear in all these expressions, which can practically be cancelled by the prefactors in the bracket in Eq. (A18). Secondly, all the $I_{0,(X,Y)}$ and $I_{n,(X,Y)}$ should be understood to be computed according to Eq. (15) or (A15) with integration limits Q_{\min} and Q_{\max} specific for that target.

On the other hand, since $|f_p|^2$ in Eq. (17) is identical for different targets, it leads to a second expression for determining m_χ [7]:

$$m_\chi|_\sigma = \frac{(m_X/m_Y)^{5/2} m_Y - m_X (\mathcal{R}_{\sigma,X}/\mathcal{R}_{\sigma,Y})}{\mathcal{R}_{\sigma,X}/\mathcal{R}_{\sigma,Y} - (m_X/m_Y)^{5/2}}. \quad (\text{A21})$$

Here $m_{(X,Y)} \propto A_{(X,Y)}$ has been assumed, and $\mathcal{R}_{\sigma,(X,Y)}$ have been given in Eq. (28). Similar to the analogy between Eqs. (31) and (A21), the statistical uncertainty on $m_\chi|_\sigma$ can be expressed as

$$\sigma(m_\chi)|_\sigma = \frac{(m_X/m_Y)^{5/2} |m_X - m_Y| (\mathcal{R}_{\sigma,X}/\mathcal{R}_{\sigma,Y})}{[\mathcal{R}_{\sigma,X}/\mathcal{R}_{\sigma,Y} - (m_X/m_Y)^{5/2}]^2} \times \left[\frac{1}{\mathcal{R}_{\sigma,X}^2} \sum_{i,j=2}^3 \left(\frac{\partial \mathcal{R}_{\sigma,X}}{\partial c_{i,X}} \right) \left(\frac{\partial \mathcal{R}_{\sigma,X}}{\partial c_{j,X}} \right) \text{cov}(c_{i,X}, c_{j,X}) + (X \rightarrow Y) \right]^{1/2}, \quad (\text{A22})$$

where I have used again the short-hand notation in Eq. (A19); note that $c_{1,(X,Y)} = I_{n,(X,Y)}$ do not appear here. Expressions for the derivatives of $\mathcal{R}_{\sigma,X}$ can be computed from Eq. (28) as

$$\frac{\partial \mathcal{R}_{\sigma,X}}{\partial I_{0,X}} = \left[\frac{F_X^2(Q_{\min,X})}{2Q_{\min,X}^{1/2} r_X(Q_{\min,X}) + I_{0,X} F_X^2(Q_{\min,X})} \right] \mathcal{R}_{\sigma,X}, \quad (\text{A23a})$$

$$\frac{\partial \mathcal{R}_{\sigma,X}}{\partial r_X(Q_{\min,X})} = \left[\frac{2Q_{\min,X}^{1/2}}{2Q_{\min,X}^{1/2} r_X(Q_{\min,X}) + I_{0,X} F_X^2(Q_{\min,X})} \right] \mathcal{R}_{\sigma,X}; \quad (\text{A23b})$$

and similarly for the derivatives of $\mathcal{R}_{\sigma,Y}$. Remind that factors $\mathcal{R}_{\sigma,(X,Y)}$ appearing here can also be cancelled by the prefactors in the bracket in Eq. (A22).

In order to yield the best-fit WIMP mass as well as to minimize its statistical uncertainty by combining the estimators for different n in Eq. (31) with each other and with the estimator in Eq. (A21), a χ^2 function has been introduced as [7]

$$\chi^2(m_\chi) = \sum_{i,j} (f_{i,X} - f_{i,Y}) \mathcal{C}_{ij}^{-1} (f_{j,X} - f_{j,Y}), \quad (\text{A24})$$

where

$$f_{i,X} \equiv \alpha_X^i \left[\frac{2Q_{\min,X}^{(i+1)/2} r_X(Q_{\min})/F_X^2(Q_{\min,X}) + (i+1)I_{i,X}}{2Q_{\min,X}^{1/2} r_X(Q_{\min})/F_X^2(Q_{\min,X}) + I_{0,X}} \right] \left(\frac{1}{300 \text{ km/s}} \right)^i = \left(\frac{\alpha_X \mathcal{R}_{i,X}}{300 \text{ km/s}} \right)^i, \quad (\text{A25a})$$

for $i = -1, 1, 2, \dots, n_{\max}$, and

$$\begin{aligned} f_{n_{\max}+1,X} &\equiv \mathcal{E}_X \left[\frac{A_X^2}{2Q_{\min,X}^{1/2} r_X(Q_{\min}) / F_X^2(Q_{\min,X}) + I_{0,X}} \right] \left(\frac{\sqrt{m_X}}{m_\chi + m_X} \right) \\ &= \frac{A_X^2}{\mathcal{R}_{\sigma,X}} \left(\frac{\sqrt{m_X}}{m_\chi + m_X} \right); \end{aligned} \quad (\text{A25b})$$

the other $n_{\max} + 2$ functions $f_{i,Y}$ can be defined analogously. Here n_{\max} determines the highest moment of $f_1(v)$ that is included in the fit. The f_i are normalized such that they are dimensionless and very roughly of order unity in order to alleviate numerical problems associated with the inversion of their covariance matrix. Note that the first $n_{\max} + 1$ fit functions depend on m_χ only through the overall factor α and m_χ in Eqs. (A25a) and (A25b) is now a fit parameter, which may differ from the true value of the WIMP mass. Finally, \mathcal{C} in Eq. (A24) is the total covariance matrix. Since the X and Y quantities are statistically completely independent, \mathcal{C} can be written as a sum of two terms:

$$\mathcal{C}_{ij} = \text{cov}(f_{i,X}, f_{j,X}) + \text{cov}(f_{i,Y}, f_{j,Y}). \quad (\text{A26})$$

The entries of the \mathcal{C} matrix given here involving basically only the moments of the WIMP velocity distribution can be read off Eq. (82) of Ref. [5], with a slight modification due to the normalization factor in Eq. (A25a)¹⁵:

$$\begin{aligned} \text{cov}(f_i, f_j) &= \mathcal{N}_m^2 \left[f_i f_j \text{cov}(I_0, I_0) + \tilde{\alpha}^{i+j} (i+1)(j+1) \text{cov}(I_i, I_j) \right. \\ &\quad - \tilde{\alpha}^j (j+1) f_i \text{cov}(I_0, I_j) - \tilde{\alpha}^i (i+1) f_j \text{cov}(I_0, I_i) \\ &\quad + D_i D_j \sigma^2(r(Q_{\min})) - (D_i f_j + D_j f_i) \text{cov}(r(Q_{\min}), I_0) \\ &\quad \left. + \tilde{\alpha}^j (j+1) D_i \text{cov}(r(Q_{\min}), I_j) + \tilde{\alpha}^i (i+1) D_j \text{cov}(r(Q_{\min}), I_i) \right]. \end{aligned} \quad (\text{A27})$$

Here I used

$$\mathcal{N}_m \equiv \frac{1}{2Q_{\min}^{1/2} r(Q_{\min}) / F^2(Q_{\min}) + I_0}, \quad (\text{19})$$

$$\tilde{\alpha} \equiv \frac{\alpha}{300 \text{ km/s}}, \quad (\text{A28})$$

and

$$D_i \equiv \frac{1}{\mathcal{N}_m} \left[\frac{\partial f_i}{\partial r(Q_{\min})} \right] = \frac{2}{F^2(Q_{\min})} \left(\tilde{\alpha}^i Q_{\min}^{(i+1)/2} - Q_{\min}^{1/2} f_i \right), \quad (\text{A29a})$$

for $i = -1, 1, 2, \dots, n_{\max}$; and

$$D_{n_{\max}+1} = \frac{2}{F^2(Q_{\min})} \left(-Q_{\min}^{1/2} f_{n_{\max}+1} \right). \quad (\text{A29b})$$

Finally, since the basic requirement of the expressions for determining m_χ given in Eqs. (31) and (A21) is that, from two experiments with different target nuclei, the values of a given

¹⁵Since the last f_i defined in Eq. (A25b) can be computed from the same basic quantities, i.e., the counting rates at Q_{\min} and the integrals I_0 , it can directly be included in the covariance matrix.

moment of the WIMP velocity distribution estimated by Eq. (13) should agree, the upper cuts on $f_1(v)$ in two data sets should be (approximately) equal¹⁶. Since $v_{\text{cut}} = \alpha\sqrt{Q_{\text{max}}}$, it requires that [7]

$$Q_{\text{max},Y} = \left(\frac{\alpha_X}{\alpha_Y}\right)^2 Q_{\text{max},X}. \quad (\text{A30})$$

Note that α defined in Eq. (5) is a function of the true WIMP mass. Thus this relation for matching optimal cut-off energies can be used only if m_χ is already known. One possibility to overcome this problem is to fix the cut-off energy of the experiment with the heavier target, minimize the $\chi^2(m_\chi)$ function defined in Eq. (A24), and then estimate the cut-off energy for the lighter nucleus by Eq. (A30) algorithmically [7].

A.3 Covariance of m_χ and $1/\mathcal{N}_m$

First, the statistical error on $1/\mathcal{N}_m$ can be given from Eq. (19) directly as

$$\sigma^2(1/\mathcal{N}_m) = \left[\frac{2Q_{\text{min}}^{1/2}}{F^2(Q_{\text{min}})} \right]^2 \sigma^2(r(Q_{\text{min}})) + \sigma^2(I_0) + 2 \left[\frac{2Q_{\text{min}}^{1/2}}{F^2(Q_{\text{min}})} \right] \text{cov}(r(Q_{\text{min}}), I_0). \quad (\text{A31})$$

For the case that one has only two data sets with different target nuclei, X and Y , one of these two data sets will then be needed for reconstructing the WIMP mass m_χ and also for estimating $1/\mathcal{N}_m$ in Eq. (17). The uncertainties on m_χ and $1/\mathcal{N}_m$ are thus correlated. Assuming that the WIMP mass is reconstructed by Eq. (31), and target $X(Y)$ is used for estimating $1/\mathcal{N}_m$, the covariance of $m_\chi|_{\langle v^n \rangle}$ and $1/\mathcal{N}_{m,(X,Y)}$ can be obtained by modifying Eq. (A18) slightly as

$$\begin{aligned} & \text{cov}(m_\chi|_{\langle v^n \rangle}, 1/\mathcal{N}_{m,X}) \\ &= \frac{\sqrt{m_X/m_Y} (m_X - m_Y) (\mathcal{R}_{n,X}/\mathcal{R}_{n,Y})}{(\mathcal{R}_{n,X}/\mathcal{R}_{n,Y} - \sqrt{m_X/m_Y})^2} \left(\frac{1}{\mathcal{R}_{n,X}} \right) \\ & \quad \times \sum_{i=1}^3 \left(\frac{\partial \mathcal{R}_{n,X}}{\partial c_{i,X}} \right) \left[\text{cov}(c_{i,X}, I_{0,X}) + \text{cov}(c_{i,X}, r_X(Q_{\text{min},X})) \left(\frac{2Q_{\text{min},X}^{1/2}}{F_X^2(Q_{\text{min},X})} \right) \right], \end{aligned} \quad (\text{A32a})$$

and

$$\begin{aligned} & \text{cov}(m_\chi|_{\langle v^n \rangle}, 1/\mathcal{N}_{m,Y}) \\ &= \frac{\sqrt{m_X/m_Y} (m_X - m_Y) (\mathcal{R}_{n,X}/\mathcal{R}_{n,Y})}{(\mathcal{R}_{n,X}/\mathcal{R}_{n,Y} - \sqrt{m_X/m_Y})^2} \left(\frac{-1}{\mathcal{R}_{n,Y}} \right) \\ & \quad \times \sum_{i=1}^3 \left(\frac{\partial \mathcal{R}_{n,Y}}{\partial c_{i,Y}} \right) \left[\text{cov}(c_{i,Y}, I_{0,Y}) + \text{cov}(c_{i,Y}, r_Y(Q_{\text{min},Y})) \left(\frac{2Q_{\text{min},Y}^{1/2}}{F_Y^2(Q_{\text{min},Y})} \right) \right]. \end{aligned} \quad (\text{A32b})$$

For the case that the WIMP mass is reconstructed by Eq. (A21), one can also modify Eq. (A22) to obtain that

$$\text{cov}(m_\chi|_\sigma, 1/\mathcal{N}_{m,X})$$

¹⁶Here the threshold energies have been assumed to be negligible.

$$\begin{aligned}
&= \frac{(m_X/m_Y)^{5/2} (m_X - m_Y) (\mathcal{R}_{\sigma,X}/\mathcal{R}_{\sigma,Y})}{\left[\mathcal{R}_{\sigma,X}/\mathcal{R}_{\sigma,Y} - (m_X/m_Y)^{5/2}\right]^2} \left(\frac{1}{\mathcal{R}_{\sigma,X}}\right) \\
&\quad \times \sum_{i=2}^3 \left(\frac{\partial \mathcal{R}_{\sigma,X}}{\partial c_{i,X}}\right) \left[\text{cov}(c_{i,X}, I_{0,X}) + \text{cov}(c_{i,X}, r_X(Q_{\min,X})) \left(\frac{2Q_{\min,X}^{1/2}}{F_X^2(Q_{\min,X})}\right) \right], \quad (\text{A33a})
\end{aligned}$$

and

$$\begin{aligned}
&\text{cov}(m_X|_{\sigma}, 1/\mathcal{N}_{\text{m},Y}) \\
&= \frac{(m_X/m_Y)^{5/2} (m_X - m_Y) (\mathcal{R}_{\sigma,X}/\mathcal{R}_{\sigma,Y})}{\left[\mathcal{R}_{\sigma,X}/\mathcal{R}_{\sigma,Y} - (m_X/m_Y)^{5/2}\right]^2} \left(\frac{-1}{\mathcal{R}_{\sigma,Y}}\right) \\
&\quad \times \sum_{i=2}^3 \left(\frac{\partial \mathcal{R}_{\sigma,Y}}{\partial c_{i,Y}}\right) \left[\text{cov}(c_{i,Y}, I_{0,Y}) + \text{cov}(c_{i,Y}, r_Y(Q_{\min,Y})) \left(\frac{2Q_{\min,Y}^{1/2}}{F_Y^2(Q_{\min,Y})}\right) \right]. \quad (\text{A33b})
\end{aligned}$$

Note that, firstly, in the above expressions we have to use $(m_X - m_Y)$ instead of $|m_X - m_Y|$ in Eqs. (A18) and (A22); for expressions with the Y target, there is an additional “ $-$ (minus)” sign. Secondly, the algorithmic process for matching the experimental maximal cut-off energies of two experiments used for the reconstruction of the WIMP mass can also be used with the basic expressions (31) and (A21). For this case and the lighter nucleus is used for estimating $1/\mathcal{N}_{\text{m}}$, the energy range of the sum in Eq. (A12) or of the integral in Eq. (A16) as the estimator for the covariance of I_n should be modified to be between Q_{\min} and the *reduced* maximal cut-off energy of the lighter nucleus.

References

- [1] G. Jungman, M. Kamionkowski and K. Griest, “*Supersymmetric Dark Matter*”, *Phys. Rep.* **267**, 195 (1996), [arXiv:hep-ph/9506380](#).
- [2] G. Bertone, D. Hooper and J. Silk, “*Particle Dark Matter: Evidence, Candidates and Constraints*”, *Phys. Rep.* **405**, 279 (2005), [arXiv:hep-ph/0404175](#).
- [3] P. F. Smith and J. D. Lewin, “*Dark Matter Detection*”, *Phys. Rep.* **187**, 203 (1990).
- [4] J. D. Lewin and P. F. Smith, “*Review of Mathematics, Numerical Factors, and Corrections for Dark Matter Experiments Based on Elastic Nuclear Recoil*”, *Astropart. Phys.* **6**, 87 (1996).
- [5] M. Drees and C.-L. Shan, “*Reconstructing the Velocity Distribution of Weakly Interacting Massive Particles from Direct Dark Matter Detection Data*”, *J. Cosmol. Astropart. Phys.* **0706**, 011 (2007), [arXiv:astro-ph/0703651](#).
- [6] C.-L. Shan and M. Drees, “*Determining the WIMP Mass from Direct Dark Matter Detection Data*”, [arXiv:0710.4296 \[hep-ph\]](#) (2007).
- [7] M. Drees and C.-L. Shan, “*Model-Independent Determination of the WIMP Mass from Direct Dark Matter Detection Data*”, *J. Cosmol. Astropart. Phys.* **0806**, 012 (2008), [arXiv:0803.4477 \[hep-ph\]](#).

- [8] H. Baer and X. Tata, “*Dark Matter and the LHC*”, arXiv:0805.1905 [hep-ph] (2008); H. Baer, E. K. Park, and X. Tata, “*Collider, Direct and Indirect Detection of Supersymmetric Dark Matter*”, *New J. Phys.* **11**, 105024 (2009), arXiv:0903.0555 [hep-ph].
- [9] R. Catena and P. Ullio, “*A Novel Determination of the Local Dark Matter Density*”, *J. Cosmol. Astropart. Phys.* **1008**, 004 (2010), arXiv:0907.0018 [astro-ph.CO].
- [10] M. Weber and W. de Boer, “*Determination of the Local Dark Matter Density in our Galaxy*”, *Astron. Astrophys.* **509**, A25 (2010), arXiv:0910.4272 [astro-ph.CO].
- [11] P. Salucci, F. Nesti, G. Gentile and C. F. Martins, “*The Dark Matter Density at the Sun’s Location*”, *Astron. Astrophys.* **523**, A83 (2010), arXiv:1003.3101 [astro-ph.GA].
- [12] M. Pato, O. Agertz, G. Bertone, B. Moore and R. Teyssier, “*Systematic Uncertainties in the Determination of the Local Dark Matter Density*”, *Phys. Rev. D* **82**, 023531 (2010), 1006.1322 [astro-ph.HE].
- [13] W. de Boer and M. Weber, “*The Dark Matter Density in the Solar Neighborhood Reconsidered*”, *J. Cosmol. Astropart. Phys.* **1104**, 002 (2011), arXiv:1011.6323 [astro-ph.CO].
- [14] P. D. Sackett, H. W. Rix, B. J. Jarvis and K. C. Freeman, “*The Flattened Dark Halo of Polar Ring Galaxy NGC-4650A: A Conspiracy of Shapes?*”, *Astrophys. J.* **436**, 629 (1994), arXiv:astro-ph/9406015.
- [15] L. Bergström, “*Dark Matter Candidates*”, *New J. Phys.* **11**, 105006 (2009), arXiv:0903.4849 [hep-ph].
- [16] R. C. Cotta, J. S. Gainer, J. L. Hewett, and T. G. Rizzo, “*Dark Matter in the MSSM*”, *New J. Phys.* **11**, 105026 (2009), arXiv:0903.4409 [hep-ph].
- [17] C.-L. Shan, “*Extracting Dark Matter Properties Model-Independently from Direct Detection Experiments*”, *Mod. Phys. Lett. A* **25**, 951 (2010), arXiv:1003.0962 [hep-ph].
- [18] M. Drees and C.-L. Shan, “*How Precisely Could We Identify WIMPs Model-Independently with Direct Dark Matter Detection Experiments*”, arXiv:0903.3300 [hep-ph] (2009).
- [19] C.-L. Shan, “*Determining Ratios of WIMP-Nucleon Cross Sections from Direct Dark Matter Detection Data*”, *J. Cosmol. Astropart. Phys.* **1107**, 005 (2011), arXiv:1103.0482 [hep-ph].
- [20] J. Engel, “*Nuclear Form-Factors for the Scattering of Weakly Interacting Massive Particles*”, *Phys. Lett. B* **264**, 114 (1991).
- [21] A. M. Green, “*Determining the WIMP Mass Using Direct Detection Experiments*”, *J. Cosmol. Astropart. Phys.* **0708**, 022 (2007), arXiv:hep-ph/0703217; “*Determining the WIMP Mass from a Single Direct Detection Experiment, a More Detailed Study*”, *J. Cosmol. Astropart. Phys.* **0807**, 005 (2008), arXiv:0805.1704 [hep-ph].
- [22] N. Bernal, A. Goudelis, Y. Mambrini and C. Munoz, “*Determining the WIMP Mass Using the Complementarity Between Direct and Indirect Searches and the LHC*”, *J. Cosmol. Astropart. Phys.* **0901**, 046 (2009), arXiv:0804.1976 [hep-ph].

- [23] C.-L. Shan, “*Determining the Mass of Dark Matter Particles with Direct Detection Experiments*”, *New J. Phys.* **11**, 105013 (2009), [arXiv:0903.4320 \[hep-ph\]](#).
- [24] E. Aprile and L. Baudis, for the XENON100 Collab., “*Status and Sensitivity Projections for the XENON100 Dark Matter Experiment*”, *PoS IDM2008*, 018 (2008), [arXiv:0902.4253 \[astro-ph.IM\]](#).
- [25] CRESST Collab., R. F. Lang *et al.*, “*Discrimination of Recoil Backgrounds in Scintillating Calorimeters*”, *Astropart. Phys.* **33**, 60 (2010), [arXiv:0903.4687 \[astro-ph.IM\]](#); CRESST Collab., R. F. Lang *et al.*, “*Electron and Gamma Background in CRESST Detectors*”, *Astropart. Phys.* **32**, 318 (2010), [arXiv:0905.4282 \[astro-ph.IM\]](#); CRESST Collab., J. Schmalzer *et al.*, “*Status of the CRESST Dark Matter Search*”, *AIP Conf. Proc.* **1185**, 631 (2009), [arXiv:0912.3689 \[astro-ph.IM\]](#).
- [26] EDELWEISS Collab., A. Broniatowski *et al.*, “*A New High-Background-Rejection Dark Matter Ge Cryogenic Detector*”, *Phys. Lett. B* **681**, 305 (2009), [arXiv:0905.0753 \[astro-ph.IM\]](#); EDELWEISS Collab., E. Armengaud *et al.*, “*First Results of the EDELWEISS-II WIMP Search Using Ge Cryogenic Detectors with Interleaved Electrodes*”, *Phys. Lett. B* **687**, 294 (2010), [arXiv:0912.0805 \[astro-ph.CO\]](#).
- [27] CDMS Collab., Z. Ahmed *et al.*, “*Results from the Final Exposure of the CDMS II Experiment*”, *Science* **327**, 1619 (2010), [arXiv:0912.3592 \[astro-ph.CO\]](#).
- [28] Y.-T. Chou and C.-L. Shan, “*Effects of Residue Background Events in Direct Dark Matter Detection Experiments on the Determination of the WIMP Mass*”, *J. Cosmol. Astropart. Phys.* **1008**, 014 (2010), [arXiv:1003.5277 \[hep-ph\]](#); C.-L. Shan, “*Effects of Residue Background Events in Direct Detection Experiments on Determining Properties of Halo Dark Matter*”, *PoS IDM2010*, 039 (2010), [arXiv:1011.2021 \[astro-ph.HE\]](#).
- [29] C.-L. Shan, “*Effects of Residue Background Events in Direct Detection Experiments on Identifying WIMP Dark Matter*”, *Int. J. Mod. Phys. D* **20**, 1453 (2011), [arXiv:1012.2625 \[hep-ph\]](#); C.-L. Shan, “*Effects of Residue Background Events in Direct Dark Matter Detection Experiments on the Estimation of the Spin-Independent WIMP-Nucleon Coupling*”, [arXiv:1103.4049 \[hep-ph\]](#) (2011).
- [30] V. Barger, W. Y. Keung, and G. Shaughnessy, “*Spin Dependence of Dark Matter Scattering*”, *Phys. Rev. D* **78**, 056007 (2008), [arXiv:0806.1962 \[hep-ph\]](#).
- [31] G. Bélanger, E. Nezri, and A. Pukhov, “*Discriminating Dark Matter Candidates Using Direct Detection*”, *Phys. Rev. D* **79**, 015008 (2009), [arXiv:0810.1362 \[hep-ph\]](#).